Who is right? Theoretical analysis of representational activities

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A theoretical discussion of possible connections between some non-conventional external representations (i.e., textual tasks of a particular format) and their corresponding internal representations (i.e., perceptions of these tasks by mathematics learners) is presented in this paper. Our goal is to analyze the interplay between external and internal representations in relation to non-conventional textual tasks of "who-is-right?" format. Such tasks involve a relatively long textual story introducing a situation that can be interpreted in (at least) two contradictory ways, which are explicitly given. The solvers of the task are required deciding which interpretation is correct and support their decision by an argument that would convince their peers. The presented theoretical analysis in terms of representations can serve as a tool for teachers and teacher educators in designing tasks specifically tailored to their students' needs.

Keywords: experientially real representations, socially shared representation, task design, controversy

Introduction

Ways in which mathematical ideas are represented are fundamental for how people learn and use these ideas (Heinze, Star, & Verschaffel, 2009). Goldin and Kaput (1996) refer to *representation* as a configuration of some kind that corresponds to, or is referentially associated with, or symbolizes something else. According to the National Council of Teachers in Mathematics, *representation* refers "to the act of capturing a mathematical concept or relationship in some form and to the form itself." (NCTM, 2000, p. 4). In other words, representations refer to both processes and products of learning and doing mathematics.

Goldin and Kaput (1996) differentiate between internal and external representations. Internal representations allude to possible mental configurations of individuals. As such, they cannot be observed directly. Rather, they can be inferred based on theoretical analysis, and when appropriate – based on behavioral clues, like gestures, speech and acts. In contrast, external representation allude to physically embodied, observable configurations such as texts, graphs, pictures, or mathematical symbols. These are in principle accessible to observation.

The theoretical discussion of possible connections between some non-conventional external representations (i.e., textual tasks of a particular format) and their corresponding internal representations (i.e., perceptions of these tasks by mathematics learners) in this paper corresponds with the Call for Papers of CERME11 TWG24 in the following way. We concern with creation, interpretation and reflection on external representations in learners' minds, with the purpose of depicting and communicating information, thinking about and developing mathematical ideas, and advancing understandings. Hence, we address the following question posed in the Call: "How can non-conventional representational activity contribute to mathematical thinking?" Specifically, our goal in this paper is to analyze the interplay between external and internal representations in relation to non-conventional textual tasks of "who-is-right?" format.

In the sections below, we present a "who-is-right?" task format, followed by analysis of examples in the contexts of algebra and geometry, with particular focus on conjectured internal representations. The conjectures, though theoretical, are supported by our experiences of enacting the presented tasks with various audiences. The concluding discussion is related to aspects of designing tasks of "who-is-right?" format, specifically to taking some conventional mathematical tasks and turning them into un-conventional ones.

The "who-is-right?" task format

An external representation of "who-is-right?" format (hereafter, WIR task) involves a relatively long textual story introducing a situation that can be interpreted in (at least) two contradictory ways. Rather than asking students to interpret the situation, the contradictory interpretations are explicitly given in the voices of two or more virtual characters. As a rule, each interpretation involves argument based on attention to some aspects of the story; otherwise, the argument supporting the interpretation must be revealed by actual solvers of the task. The actual solvers of the task are required deciding which interpretation is correct and support their decision by an argument that would convince their peers.

Such tasks have several representational characteristics. First, the textually presented situation alludes to experientially real world of the potential solvers of the task (Gravemeijer & Doorman, 1999). That is, the situation is rooted in solvers' perception of experientially real, everyday-life phenomena. Second, the task is presented in everyday language, which might be ambiguous once referring to mathematical objects or mathematical objects in disguise. Third, the task does not require solving a textbook-like mathematical problem, at least not explicitly. Instead, two contradictory claims about the situation are presented. The learners are asked to take a stance and defend it. In order to be able to do so, the learners need to build two internal representations of the situation by taking into account each of two contradictory claims. This activity has the potential to advance their ability to overcome the limitations of the experientially real (but subjective) representations, towards building a shared representation based on mathematically valid argument that can convince the other learners when non-mathematical argument fails to do so. Forth, the mathematically valid answer to the question "who is right?" may be not only of the "one of the characters is right" format but also of non-conventional formats such as "both are right", or "both are wrong", or "it depends". Such options potentially adds an element of surprise to the task. Fifth, WIR tasks can be sequenced so that dealing with the next task requires not only a comparison between two interpretations of the given situation but also comparison of a new situation with the previous one.

Several WIR tasks have been used in previous studies as research instruments. For example, scholars (Healy & Hoyles, 2000; Buchbinder, 2010, Buchbinder & Zaslavsky, 2013) have used such tasks for revealing the student conceptions of proof and examples. In particular, Buchbinder and Zaslavsky (2013) designed a WIR task based on a claim "For every natural $n, n^2 + n + 17$ is a prime number" plus two responses to this claim by virtual student-characters. The first student-character argued that the claim is correct because she checked its validity for the first 10 integers, and the second student-character claimed that the statement is false because $n^2 + n + 17$ is not prime for $n = 16$. Then 12 high-school students were exposed to the task and asked to determine who is right and why. The study revealed interesting inconsistencies in students' perceptions of proof.

Another WIR task was used in a teaching experiment conducted by Koichu (2012) with a group of pre-service teachers. The teachers were exposed to two pictures: a picture of the famous Penrose triangle and a picture representing a sculpture of the Penrose triangle taken from an angle that created an impression that the Penrose triangle can be made as a real 3D object. The students were required deciding whether the Penrose triangle indeed could be a 3D object or proof otherwise. This study resulted in description of a particular set of instructional conditions in which mathematical defining and proving could be intellectually necessitated for the students.

In spite of the fact that in both tasks one of the included responses was intuitively more appealing than the other one, the tasks proved themselves as triggers for evoking uncertainty in their solvers as well as useful tools for revealing the students' (mis)conceptions and reasoning. These tasks stimulated our interest in further developing WIR representational activities as research and teaching tools that would correspond to curriculum-determined mathematical topics. In the next section, we present two WIR activities, one in algebra and another in geometry, and discuss their common features as well as subtle differences inherited in their design.

Examples and analysis

The WIR tasks presented in this section are designed for the use with middle-school students. The forthcoming discussion of internal representations associated with these tasks is based on our experience of enacting and discussing the tasks with two groups of in-service teachers.

Example 1: Subtle matters related to percentages

Figure 1 presents the first task in the sequence of three tasks.

<table>
<tr><td colspan="2">Towards the end of the summer holidays, suitcases are sold with a 30% discount. The final price of a suitcase includes VAT of 17%.

In calculation of the price, one can first take into account the discount and then calculate VAT or otherwise. It appears that a buyer and a tax collector have different opinions on this matter.</td></tr>
<tr><td>Advice from Buyer: I think that one needs first to calculate the price including VAT and then the discounted price. In this way, the discount would apply to a greater number and the final price would be lower.</td><td>Advice from Tax Collector: I think that one needs first to calculate the discounted price and then to add VAT. In this way, the sum of VAT would not be influenced by the discount and the final price would be higher.</td></tr>
<tr><td colspan="2">1. What do you think?
2. How would you convince a peer who disagrees with you?
3. How can we decide who is right?</td></tr>
</table>

Figure 1: The first task (discount and tax)

We now outline a conjectured scenario of dealing with this task. The student reads the story and decides who is right (a buyer or a tax collector) based mainly on the emotionally loaded process of taking the perspective of a buyer or of an authority. It was our intention to make the buyer's and the tax collector's responses more or less equally plausible for the solver. The process of the task comprehending requires the students to create competing internal representations of the situation, but at the individual level, one of two internal representation eventually prevails, and the student convinces herself that "a buyer is right" or "a tax collector is right". At the collective level, the class splits (always worked in our teaching), so that each student can see that his or her opinion is not the only possible one. At this stage, many students cannot think of an argument in support of the opposite opinion, so the split is perceived as a surprise. When asked by the teacher to convince the opponents, the students attempt to explicate their internal representation and produce explanations in everyday language often accompanied by gestures. Then the students are surprised even more when they find out that these explanations are incomprehensible or perceived as too vague for their counterparts in order to be convincing. A collectively shared need to find a way of getting to the common ground arises. At this stage, an option to employ mathematical apparatus emerges, usually as a spontaneous suggestion of one of the participants. This is probably because of the general context: after all, the discussion occurs at a mathematics lesson. Some meta-level questions arise with the help of the teacher. For example, what argument can be convincing? How can the buyer argument and the tax collector argument be expressed mathematically?

Next, the teacher suggests the students to attempt to mathematize the situation, in the hope to construct a shared (external) representation that would be convincing for all. The students begin using mathematical language, first in order to test the correctness of their initial claims and then in order to assess their claims as well their opponents' claims. In this process, the classroom map of agency changes, and the teacher does not act as a source of knowledge about the correct solution but as a facilitator of establishing the new discursive rules (Sfard, 2002), and as a mediator of the discussion. In this way, the teacher stops being an authority whose role is to approve or disapprove the student solutions, but becomes an authority who can support or disregard (for example, by re-voicing) types of arguments produced by the students. This situation promotes the individual student argumentative talk based on gradually established rules (Hershkowitz, Tabach, Rasmussen, & Dreyfus, 2014).

Eventually, the surprising to many students resolution of the given situation – the buyer method and the tax collector method lead to the same final price – may ease the emotional tension in class. This would be a good moment to present the next task of the sequence (Figure 2).

Presenting this task second, one cannot but compare it with the first task, which has just been solved. In this context, the individual may create an internal representation of the task by attending to the following parallels: decreasing the length of one side is the counterpart of the discount, and increasing the length of the other side is the counterpart of the VAT. This may overshadow for the students an important difference between the external representations of the tasks. It is now not about the multiplication operations performed sequentially, where the second operation is applied on the result of the first operation, hence the final answer does not depend on the order. In the second task, each of two multiplication operations is applied to the same initial number, and then the results of these two operations are multiplied to find the area of a new shape, hence the final result may change.

The area of any square equals to the area of a rectangle built from that square by increasing one side by 10% and decreasing the perpendicular side by 10%. Is it so?	
Michael: Yes. The square sides are of equal length. Therefore, adding 10% and cutting 10% of the side lengths imply that what you add and what you cut is the same.	**Josef**: No. If the side length is 10 cm, the area is 100 cm^2. The rectangle will have 9 and 11 as side lengths, and the area will be 99 cm^2.
1. What do you think? 2. How would you convince your peer who disagrees with you? 3. How can we decide who is right?	

Figure 2: The second task (changing the area)

In this case, using the same 10% change for the adjusted sides of the square may add to the internal representation that leads to the (wrong) conclusion that "nothing changes". For these reasons, reading Josef's explanation may serve as an eye-opener, in line with the counterexample for $n = 16$ in the task from the Buchbinder and Zaslavsky (2013) study, as presented above. The Josef calculations are easy to follow and validate. Hence, we expect that the discussion of the "who-is-right?" question would rather be quick, and that the discussion of the question "how the first and the second tasks are different?" would be longer. Then the discussion might develop, again, with the help of the teacher, in an epistemological direction. Namely, the class may discuss what can safely be concluded from just one counterexample and would general argument make the Josef example more convincing.

Our suggested sequence does not stop here. The third task is ready to be presented (Figure 3).

The perimeter of any square equals to the perimeter of a rectangle built from that square by increasing one side by 10% and decreasing the perpendicular side by 10%. Is it so?	
Ron: No. We just solve this problem and saw that it would not be the same.	**Gal**: Yes. Two sides became larger and two become smaller. The adds and cuts are of the same size; hence, the perimeter will not change.
1. What do you think? 2. How would you convince your peer who disagrees with you? 3. How can we decide who is right?	

Figure 3: The third task (changing the perimeter)

Reading the third tasks may immediately raise a déjà-vu feeling of we have already solved this one, and, indeed, Ron's claim gives room for this feeling. Yet, reading Gal's argument may lead the learners to re-consideration of Ron's stance. We believe that for this task to be the third in the sequence, middle school students may turn to the search for (external) mathematical representations of the situation quicker than in the previous two tasks. And once more, surprise would await them (after all, Gal is right!), which may evoke a hot discussion of how this task is different from the previous one. From our experience, elaboration on the differences between the external representations of all three tasks in tacit association with students' internal representations, which gradually emerge in the classroom discourse, is a non-trivial endeavor. This was true even for the in-service mathematics teachers, who were exposed to the sequence, and even for those of them who were familiar with a variation of the first task. All the teachers acknowledged the plausibility of the above scenarios and expressed the wish to use the sequence with their students.

Example 2: Dynamic loci of points

A task presented in Figure 4 preserves the WIR format, but differs from the previous example in (at least) three important respects: (1) it concerns geometry; (2) it requires reconstruction of two different arguments rather than comprehending the given arguments, hence the questions to the solvers are not as in Example 1; (3) the answer to the task is of "it depends" format.

<table>
<tr><td colspan="2">Einat and Sarah like touring and exploring. One day they found an old wooden box containing a note, written, apparently, by pirates. The note said: "There is no choice, we retreat. The treasure is too heavy and we are forced to hide it. The treasure is in 200 steps from a segment connecting the tallest eucalyptus tree and the tallest maple tree of this area, and in 300 steps from the midpoint of this segment."

Einat and Sarah found the eucalyptus and the maple, but then argue what to do next.</td></tr>
<tr><td>Einat: There is no chance to find the treasure. It just does not exist.</td><td>Sarah: Let's try. With small effort, we have a chance to find the treasure.</td></tr>
<tr><td colspan="2">1. What might be the Sarah argument?
2. What might be the Einat argument?
3. Unver which conditions could Einat or Sarah be right?</td></tr>
</table>

Figure 4: The treasure task

With no exception, the solvers begin from drawing a sketch (external representation), as the one presented in Figure 5.

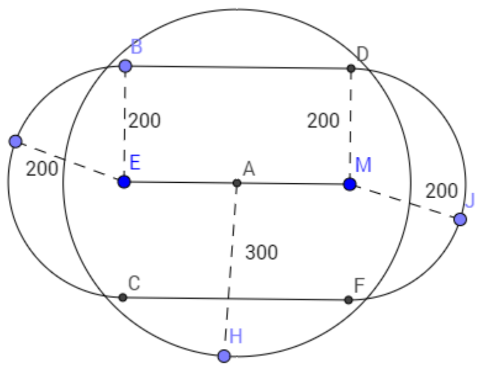

Figure 5: A sketch for treasure task (E stands for the eucalyptus, M stands for the maple)

At first glance, the sketch supports Sarah's opinion, because the stadium-like shape (represents the locus of points at the distance of 200 steps from segment EM) and the circle (represents the locus of points at the distance of 300 steps from the middle of EM) intersect in exactly four points. Therefore, according to the sketch, the treasure must be in one of these points, and checking only four points is a feasible task. Of note is that creating the above sketch has been proven as difficult for the solvers. With in-service teachers, the most interesting discussions revolved around two questions: How can the distance between a point and a segment be defined?" and "What is locus of points?"

The most challenging part of the task is to re-construct the argument that may underlie the Einat opinion. The tentative (but supported by practice) scenario of doing the task relies on the following assumption. Any external representation of the task situation by means of a drawing relies on an internal representation that includes an assumption about the length of segment EM. Most of the learners intuitively assume that it is longer than 200 steps, as in Figure 5. However, if $EM = 200$, the locus of points described in the situation would consist of exactly two points. Furthermore, if EM is less than 200 steps, the locus becomes an empty set, and in this case Einat is right. Since the story does not include information about the length of EM, but only the information that Einat and Sarah found the trees, each of them can be right. Hence, the answer to the task is "it depends".

The readers are invited to explore the situation by means of a dynamic GeoGebra applet at https://www.geogebra.org/m/n5mRrtMf (consider varying the position of point B at the applet).

Of note is that whether or not a dynamic external representation of the task by means of GeoGebra is used, dealing with the task requires the learner to create a dynamic internal representation of the situation. This is an additional point of difference: in Example 2, the dynamic representation might be continuous, and dealing with the change in Example 1 might require creation of a discrete series of static representations.

Discussion

On the face of it, turning "conventional" tasks into WIR tasks is simple. Instead of concluding a textual description of a situation by an open-ended question (e.g., "would the price change?" or "make a map and denote the treasure"), an additional layer of two contradictory arguments by virtual characters is added. Yet, we seek arguments that can be perceived by the learners as plausible, so that the class would split between favoring one argument over the other. Hence, designing worthy WIR tasks is not as simple as it may seem.

As mentioned in the introduction, any analysis in terms of internal representation cannot be conclusive just because these are not directly accessible. Accordingly, we do not assume that the above-presented scenarios are realistic or that we can realistically describe what is going in the minds of the solvers. Yet, the value of our analysis is in that it can serve as a design heuristic for teachers and teacher educators, like in thought experimentation, when designing WIR tasks.

Our hypothetical analysis of the interplay between internal and external representations was mostly at the level of an individual solver, and thus is in line with the claim made by Rittle-Johnson and Star (2011): "[c]omparison is a fundamental part of human cognition and a powerful learning mechanism (p. 221)". In addition, we approached the interplay between the individual and the social in implementation of the WIR tasks. In sum, we have argued that a "who-is-right?" task can present the solvers with opportunities to engage with classroom-level argumentative talk based on an emerging need to resolve conflicting opinions expressed by individuals, an hence to be a step towards constructing some socially shared internal representations of the task.

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