HISTORY OF MATHEMATICS AS A COMPONENT OF MATHEMATICS TEACHERS BACKGROUND

Ph.D. thesis
Part I

by
Abraham Arcavi

Submitted to the Scientific Council
Weizmann Institute of Science
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Contents

Chapter 1

INTRODUCTION

This thesis deals with history of mathematics as a component in teacher training. It describes a project which included a review of the literature, the assessment of the local situation (needs,status of the study of the history of mathematics, etc.), an a priori definition of the variables (what history to study, in which way, specific goals, etc.), historical research of the topics to be studied and the subsequent development of original learning materials, and their implementation in various teacher training courses.

The Case for the History of Mathematics

History of mathematics and its value in mathematical education in general, and for mathematics teachers in particular, has been advocated for a long time.

For example, in its consideration of the training of mathematics teachers, the Mathematical Association of America (1935) distinguished between minimum and desirable requirements. History of mathematics was recommended for inclusion in the first category. Almost fifty years later, twenty four distinguished mathematics educators answered the question "what are the most important books for a secondary teacher to read?". The history of mathematics was the subject most recommended (Leake, 1983).

Many other individual authors from various countries have also stressed the importance of the history of mathematics using a variety of arguments to support their claims. These arguments can be classified as follows.

1. <u>Better learning and understanding of mathematics</u>

"I am sure that no subject loses more than mathematics by any attempt to dissociate it from its history." (Glaisher, 1890).
Jones (1969) maintains that a "sense of history of mathematics coupled with an up-to-date knowledge of mathematics and its uses, is a significant tool in the hands of the teacher who teaches 'why'".
A similar, and maybe stronger, argument can be found in a report of the British Ministry of Education (1958), in which a chapter is devoted to <u>The History of Mathematics and its Bearing on Teaching</u>. "The teacher who knows little of the history of mathematics is apt to teach techniques in isolation, unrelated either to the problems and ideas which generated them or to the further developments which grew out of them."

In summary, the argument is, that knowledge of the history of mathematics can provide a deeper understanding and insight, relevant to the teacher in his/her professional life.

2. The image of mathematics and mathematical activity

"...There is no sense of history behind the teaching, so the feeling is given that the whole system dropped down ready-made from the skies, to be used by born jugglers."(Barzun, 1969). In fact, "...doubts have existed and continue to exist, and mistakes, paradoxes and controversies have special claim for consideration."(Ministry of Education Pamphlet, 1958). In other words, an appreciation of the nature of mathematics as a living and dynamic subject, and of mathematical activity as a human endeavor, cannot be fully grasped without the knowledge of the history of mathematics. "Probably the greatest usefulness of the history of mathematics is due to the fact that it puts more life into the study of this science. It changes the concepts of mathematics from the domain of statics to that of dynamics."(Miller, 1916). It may well be that teachers who have some knowledge of the history of mathematics, are in a better position to transmit a more balanced picture of the nature of mathematical activity to their students, rather than the bare facts of the syllabus.

3. Improvement of attitudes towards mathematics

History of mathematics may serve a double purpose: it can motivate, refresh and revive the teacher's own interest in the subject he/she teaches, and also provide him/her with materials to "spice" the "teaching with conversation and anecdotes" (Struik, 1980); because, "does not a rock appeal more to our imagination when we realize that it has a story?"(Barwell, 1913).

History may also give teachers and their students a healthier perspective of their own difficulties. "When a student hears how long it took for the concept of negative numbers to develop and become accepted, used and understood, he does not feel quiet so concerned that the concept didn't come to him easily."(Jones, 1957).

4. A plausible parallelism between the historical development of concepts and human development.

"Ontogeny recapitulates phylogeny" meaning in our context, "...the history of each individual development is a brief compendium of the history of the race... the sound method of instruction is to let the student travel in his quest for knowledge, roughly over the same path by which his fathers arrived, - roughly, only, because life is short, and there were quagmires in which our fathers floundered for many centuries." (Barwell, 1913).

Barwell also outlines a parallelism between the development of mathematics, on the one hand, and the cognitive development of each individual, on the other.

In fact, genetic epistemology deals extensively with this issue, also analyzing cases which are considered to be exceptions, such as the "genetic paradox of geometry" in which the historical order of evolution is the reverse of the succession of psychogenetical stages (Piaget, 1975). The parallelism argument is not undisputed, but taken at its face value, can be interpreted in a way that comes close to the previous category and complements it. Thus "The early history of the mind of men with regard to mathematics leads us to point out our own errors; and in this respect it is well to pay attention to the history of mathematics."(De Morgan, 1865). Which, in other words, means that the history of the development of mathematical concepts, with its progress, but also with its periods of stagnation, concepts which have disappeared, blind alleys etc., may provide the teacher with another source of understanding (and tolerance) of student errors and misconceptions. "The fact that many important concepts entered so slowly into the intellectual life of the world and met with much opposition, is full of meaning to those who are meeting these concepts for the first time or are trying to teach them to others."(Miller, 1916).

In summary, the above arguments present reasonable theoretical support for the study of the history of mathematics, both in pre- and in-service teacher training. However we did not find reports of controlled or semi-controlled experiments that examine the claims made. Furthermore, except for the "conventional" lecture course or textbook, we did not find historical materials specifically directed and relevant to the training of mathematics teachers. The creation of such materials requires historical research blended with educational considerations.

In the following chapters we shall describe the different stages of an experiment which included the assessment of our target population background, historical research directed to the development of original learning materials for teachers, and their implementation.

Chapter 2

ANALYSIS OF TARGET POPULATION BACKGROUND

The literature review in the previous chapter, gives some theoretical support for the development of history of mathematics material for teachers. But, in addition, we had other reasons, based on the observation of our teacher population. We were aware of the fact that, many teachers did not have the necessary minimum mathematical background which might be considered desirable for their efficient functioning in the classroom. Therefore, we were also looking for ways of providing another opportunity of learning mathematics, which were preferably novel to the teacher, so preventing a feeling of deja vu.

In order to clarify and quantify the situation, we undertook an assessment of

a) previous teacher knowledge in the history of mathematics and attitudes towards its study;

and some components of their professional background mainly concerning

b) teacher perceptions of certain aspects of the nature of mathematics, relevant to the curriculum they teach, or they are about to teach (junior-high school), and

c) mathematical knowledge of certain topics in the curriculum, or related to it.

In this chapter we shall describe aspects of the assessment undertaken with different groups of teachers, some of the items in the questionnaires administered, with the analysis of their responses, and the conclusions drawn from the assessment. The assessment was undertaken at a preliminary stage (populations A and B below) and continued with the first trial version of the materials (Populations C and D). The preliminary stage formed the basis on which we formulated the objectives, constructed the materials and implemented them.

Population

The assessment of the local situation was carried out on groups of prospective and in-service teachers, who were teaching or, were about to teach, in junior-high school. These groups are part of the target population for which the Mathematics Group of the Science Teaching Department (at the Weizmann Institute of Science) works.

The groups assessed were:

Population A: 60 prospective teachers from six classes, each class from a different teacher college, all of them during the last month of their formal studies.

Population B: 84 mathematics teachers who participated in in-service courses, most of them without university degree.

Population C: 36 teachers who attended the first in-service workshop held on the history of negative numbers (the workshop, together with other implementation activities, will be described later). Most of the teachers in this group were graduates and/or experienced teachers who had attended various in-service workshops in the past.

Population D: 56 teachers who attended an in-service workshop held on the history of irrational numbers (the workshop, together with other implementation activities, will be described later). Most of the teachers in this group had the same characteristics as in Population C.

The questionnaires

The questionnaires were different for each population. The questionnaire administered to population A included mathematical, attitude and general background questions (see Appendix 1). For this population, with the agreement of the institutions concerned, the questionnaire was presented as part of the formal feedback requirements.

For populations B, C and D (who participated in in-service courses on a voluntary basis), we contented ourselves, at this stage, with attitude or general background questions only. These questions were multiple choice, and those on attitude were statements, in which the respondent was asked to indicate the degree of agreement on a Lykert type scale: strongly disagree (1) - strongly agree (6).

Analysis of the responses

a) Previous knowledge and attitudes towards the study of history.

This aspect was reviewed in all the groups.

In population B the questions related to

i) previous knowledge of topics in the history of mathematics

ii) teacher evaluation of the role of the history of mathematics in their own personal background; and

iii) their willingness to participate in in-service workshops on topics from the history of mathematics.

The vast majority (83%) declared that they had never learned any history of mathematics, but agreed that the subject was important (71%), or at least quite important (an additional 26%). Most of them (81%) declared an interest in participating in a workshop on the subject. Thus, for this population, history was an almost unknown subject, but there was a generally favorable predisposition towards its study.

The status of the study of the history of mathematics in population A (prospective teachers) was tested by one question. Only a few (15%) had attended either a course or a workshop on the history of mathematics.

In population C (graduate and/or experienced teachers), 36% of the respondents declared that they had studied history of mathematics and, most of these had attended a course on the subject; 47% declared that they had not learned any history.

A similar picture was obtained for Population D. The relatively high percentage of people in these groups, who had studied history, can be explained by the fact that most of them had an academic background (universities in Israel or abroad), and therefore the chances that a history course was available to them, were greater than in a teacher college. The fact that these teachers came to a workshop on the history of mathematics, would suggest that the interest in history persisted after their previous studies. Teachers in Population C indicated, not only an interest, but also their view that history is very important (29%), important (an additional 59%), or at least quite important (an additional 12%). None of them indicated that it is not important.

b) Teacher perception of certain aspects of the nature of mathematics.

The following questions were part of the questionnaire for Population C.

- In your opinion, when were the negative numbers mathematically defined?
- In your opinion, when did most mathematicians make free use of negative numbers?

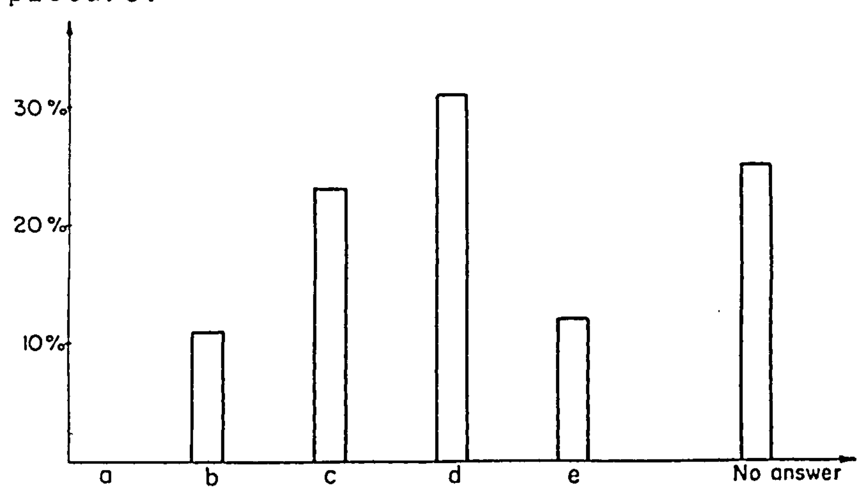
The options given for both questions were:

a) Before the Common Era (Greeks, Babylonians, etc.)
b) Early Middle Ages (Hindus, Arabs)
c) Between 1300-1600 (Europeans)
d) Between 1600-1800 (Europeans)
e) Between 1800-1900 (Europeans).

Many respondents abstained (23% for both questions and an additional 13% for one of the two), which may indicate lack of knowledge. In fact, at the time the questionnaire was administered, many teachers reacted verbally by saying that they did not know, and then they were encouraged to guess as intelligently as possible. Our purpose was not to see whether their guesses were correct or not, but to investigate a much more interesting issue, which is the relative chronological order attributed to the definition and the free use of negative numbers.

Only 11% of the sample gave the correct order: the period of the free use preceding the period of definition. For another 36% the mathematical definition preceded the free use, indicating an incorrect view of mathematics and mathematical activity. An additional 17% gave the same period for the definition and the use. Here it is not clear whether they thought that the events took place simultaneously, or that the definition preceded the use, or the reverse.

From the responses to the first question we get the following picture.



As can be seen, only a few teachers indicated the period 1800-1900. This may not only suggest lack of historical knowledge, but also a lack of understanding of the meaning of a formal mathematical definition. It is possible that the teachers regarded the intuitive (and also model related) introduction to negative numbers, similar to those appearing in the textbooks they use, as a mathematical definition. If that is indeed the case, this would provide an explanation for the "definition-use" order manifested in teachers' answers described above. In either case, there is some point to correcting their misconceptions.

Somewhat similar results were obtained from the pre-test administered to the participants that came to a workshop on the irrationals (Population D). Here also it was our intention to collect some data about previous knowledge on historico-mathematical issues on irrationals.

The question: "In your opinion, when was the first time that the concept of irrationality arose?

a) Before the Common Era (Greeks, Babylonians,etc.)

b) Early Middle Ages (Hindus, Arabs)

c) Between 1300-1600 (Europeans)

d) Between 1600-1800 (Europeans)

e) Between 1800-1900 (Europeans)."

was answered correctly (item a) by about 70% of the participants. But in a further question, in which they were asked to order chronologically the appearance of the three concepts - negative numbers, decimal fractions and irrationals - about 55% of the teachers indicated that decimal fractions preceded the irrationals and an additional 10% did not answer at all.

This was not only an indication of lack of historical knowledge (about the development of decimals), but also could be an indication that the concept of irrationality, although associated (by most of them) with the Greeks, is conceived as relying upon the decimals, and is not connected with geometry (commensurable and incommensurable segments) as it occured historically. It may well be that the intrinsic "logical chronology" of the curriculum, in which the teaching of decimal fractions far precedes the teaching of irrationals, is automatically assumed to be the historical development.

In this pre-test we also asked: "Do you know any mathematical definition of irrational numbers? Yes/No. If yes, which?". About 71% of the participants responded yes, but almost all of them (90%) quoted one of the "definitions" from the textbook they use; e.g. "a number that cannot be expressed as a quotient of two integers" or "a number whose decimal part is not periodic and has infinite number of digits". Only 5% mentioned, for example, Dedekind cuts.

We found further evidence to support the view that many teachers do not distinguish between (formal) mathematical and intuitive definitions. In the questionnaire (Appendix 1) administered to population A, a series of statements were given, for which the respondents had to indicate their degree of agreement. The following are two of the statements.

> - Mathematical concepts can be used even when their mathematical definition is not known.
>
> - Mathematical textbooks for junior-high school, in general, give the mathematical definition of every new concept.

The prospective teachers were inclined to disagree with the first statement (mean=2.9 on the Lykert scale) and to agree with the second (mean=4.6), whereas, in general, the school textbooks give only an intuitive description of many of the concepts presented. This confusion regarding the nature of definitions is not surprising, since 58% of the same sample declared (in response to other questions testing previous knowledge), that they had not learned the definition of negative numbers as equivalence classes of ordered pairs of natural numbers (an additional 18% did not answer), and 80% declared that they had not learned the definition of irrational numbers as Dedekind cuts (8% did not answer).

In order not to give the wrong impression, we would add that the above should not be taken to mean that we expect junior-high school teachers to remember the details of the formal mathematical definition, but at least they should be aware that

a) the description given in junior-high school texts of some concepts, is not a formal mathematical definition, and

b) such a definition exists. We would also expect them to appreciate that, the former is guided by didactical

considerations, and can be justified in terms of the mathematical immaturity of the target student population.
These results lead us to the conclusion that, it would be a desirable feature of any materials on the history of mathematics, to present the search for a formal mathematical definition of a particular concept and, thus, to give a proper picture of mathematical activity, as well as to motivate the necessity for the definition itself.
Students in colleges and universities are very often presented with mathematical definitions, divorced from the context which gave rise to them, and which can contribute to the feeling of logical necessity for such definitions. History seems to be a natural vehicle for fostering this feeling, since "The history of mathematics is a bountiful source of such examples showing ways in which previous generations have experimented and discovered the need for formal mathematical structures." (Meserve, 1983).

Another aspect related to the nature of mathematics and mathematical activity, will be exemplified by the following question from the population A questionnaire.

A teacher presented the following rule to the class.

$$a^m \times a^n = a^{m+n} ,$$

with the justification that

$$a^m = \underbrace{a \times \ldots \times a}_{m \text{ times}} \quad \text{and} \quad a^n = \underbrace{a \times \ldots \times a}_{n \text{ times}}$$

implies

$$a^m \times a^n = \underbrace{a \times \ldots \times a}_{m+n \text{ times}} = a^{m+n} .$$

He then continued: "If we set m=0,

we obtain

$$a^0 \times a^n = a^{0+n} = a^n .$$

Whence $a^0 \times a^n = a^n$

whence $a^0 = 1$ ".

Upon which one student asked:

"Is that a proof that $a^0 = 1$?"

What would your answer be?.Justify.

About 56% gave the correct answer, which is "no", 41% answered "yes".

An analysis of the explanations given, indicate that not all of those who said that the above is not a proof, understood why. In fact half of them, proceeded to give an incorrect explanation; for example "no, the proof is..." and then gave an identical argument but based on $a^m : a^n = a^{m-n}$.

Others said that it was not a proof because the argument is circular: it shows that $a^0 = 1$, based on $a^0 = 1$. Other explanations included: "this is only an example", "it is not general".

The correct answer ($a^0 = 1$ is a definition, and the above presentation can only provide motivation for the definition, since m,n are by implication natural numbers) was given by about 25% of the whole sample.

There was an interesting variety of answers given by those who maintained that the presentation is, in fact, a proof. For example,

"all the steps are correct",

"the conclusion is derived from a universal rule for a particular case".

The responses to the questions reviewed in this section would indicate that, a considerable number of teachers have an incorrect perception of certain aspects of the nature of mathematics and mathematical activity. Thus, we have seen that there is confusion about the chronological order of definition and use of a concept, which relates to the teachers' background and not directly to their classroom practice. On the other hand, the question on the zero exponent, directly relates to the classroom, and also to the concept and role of definition in mathematics.

c) Teacher knowledge of some topics in the curriculum.

As an example we bring a question (item 2) from the population A questionnaire.

Seven numbers were given, and the respondents were required to mark those which are irrational.

About 60% of the respondents had two or more errors. The most common error was to indicate 22/7 as irrational.

Clearly, the question does not expect of the teachers to be able to define irrational numbers, or even to be aware of a definition of irrational numbers, they are merely being asked to identify an irrational number when they see one. In spite of the fact that they are about to teach the topic, some of them are not sure, for fairly common numbers in the curriculum, whether they are irrational or not.

Conclusion

Some conclusions can be drawn from the data presented. First of all, we have some support for the universal claim that "...there is a strong indication of enormous gaps in student's knowledge of the history of mathematics. School courses rarely seem to include anything, university courses appear to do little better and we are left with a large amount of ignorance. This lack of knowledge is a cause of concern, especially for those who are about to become teachers."(Cornelius, 1984).

We also see that there is a gap between the a priori (theoretical) importance attributed to the history of mathematics and its (neglected) status in teacher training programs. Further, there are some aspects which can be reasonably demanded as part of a desirable mathematical background, that teachers lack. A historical presentation seemed to us, in addition to the arguments from the literature, a rich environment in which these aspects can be approached.

Chapter 3

HISTORICAL REVIEW

In this chapter we shall describe the historical review of the negative and irrational numbers, undertaken as part of the necessary preparation of the learning materials. These topics were chosen because of their importance in the junior-high school curriculum. Both the historical review and the subsequent development of the materials are a major part of this thesis.

<Technical note: although the learning materials - brought in Part II of this thesis - have not yet been described, we shall quote paragraphs from them, as necessary to illustrate the argument. Reference will be made in the following form: Peacock, n.n., which means the worksheet whose title is Peacock in the sequence on negative numbers. The sequence on the irrationals will be denoted by i.n.>.

The historical review included:

-- secondary sources,

-- primary sources,

and subsequently,

-- the creation of a framework for the story of the conceptual development.

The Negative Numbers

The major part of this chapter will be devoted to the discussion of the conceptual history of the negative numbers, which is almost undocumented in the secondary sources, and even the little that exists is not error free. By lack of documentation we mean the almost complete omission of the topic in works devoted to the number concept, and the very small amount of information in more general histories, concerning the different stages of the development of the negative number concept and its arithmetic (as it will presented later in this chapter).

We classified the errors found in the secondary sources, as follows.

Factual historical error. For example, Bourbaki (1960) states that the pure formal mathematical definition of negative numbers was given by Bombelli, at the end of the 16th century. In our research we found no traces of such a definition.

Implicit misrepresentation. For example, the assertion found in many secondary sources, that Euler believed that negative numbers are greater than infinity. In our research we found a source by Euler in which he states precisely the opposite. If Euler did not say it, it is, in fact, a factual error. And even if, somewhere - which we did not find - he did say it, it misrepresents Euler's view because it does not take into account all his writings on the subject.

Interpretation. For example, to confuse the meaning of the minus sign as an operation and as a sign of a negative quantity. This led to the assertion that the Greeks, for instance, "accept implicitly what later was called the law

of signs: the product of any two numbers with like signs is positive, the product of any two numbers with unlike signs is negative." (Gardner, 1977). Since Greeks do not seem to have dealt with negatives at all, we regarded the above as an error of interpretation.
A full discussion of the coverage of the topic in secondary sources, together with further historical errors which we found, are brought in Appendix 3, and are intended as a further contribution to the study of the history of negative numbers.

History of negative numbers in primary sources and a framework for their conceptual development

In this section we outline a suggested framework for the conceptual history of the negative numbers, as it emerges from the source materials we traced.
In the construction of the framework, one principle taken into account was, the distinction between the "practical use" of negative numbers and the "satisfactory explanation upon which the correct usage should be based" (Miller, 1933), and this in the light of the fact that, until the latter half of the 19th century, we have not found any mathematical definition of negative numbers.
Therefore, in the history of negative numbers until the formal mathematical definition was given, we make the

following distinction.

(i) Discussion of the concept of a negative number or quantity. This includes, among other things, the negatives arising in the solution of problems, i.e. negative roots of equations and their interpretation, and also the concept of order.

(ii) Discussion of the four operations with negative numbers, of which multiplication is the most interesting.

The interaction between these two will also be of interest. For example, to what degree, if any, does an author link his discussion of the concept of negative numbers to his discussion of the operations.

(i) <u>The concept of negative numbers</u>.

According to many historians, it seems that the early history of negative numbers is related to their acceptance or rejection as roots of equations (although there may be some exceptions: according to Tropfke (1980), the Babylonians had some idea of "subtractive numbers"). For example, in the process of solving an equation we find, in the third century A.D., in Diophantus (Heath, 1964)

> Their difference is $x^2 - 4x = x(x - 4)$, and the usual method gives $4x + 20 = 4$, *which is absurd*, because the 4 ought to be some number greater than 20.

Also in the Middle Ages, negatives are mainly related to their appearance as roots of equations; for example, in Bhascara - 12th century - (Colebrooke, 1817 - the point above a numeral is the equivalent of our minus sign).

> Example: The fifth part of the troop less three, squared, had gone to a cave; and one monkey was in sight, having climbed on a branch. Say how many they were?[3]
>
> Here the troop is put *ya* 1. Its fifth part is *ya* $\frac{1}{5}$. Less three, it is *ya* $\frac{1}{5}$ *ru* $\frac{\dot{1}5}{5}$. This squared is *ya v* $\frac{1}{25}$ *ya* $\frac{\dot{3}0}{25}$ *ru* $\frac{225}{25}$. With the one seen ($\frac{25}{25}$), it is *ya v* $\frac{1}{25}$ *ya* $\frac{\dot{3}0}{25}$ *ru* $\frac{250}{25}$. This is equal to the troop *ya* 1. Reducing these sides of equation to a common denomination, dropping the denominator, and making equal subtraction, the equation becomes
>
> *ya v* 1 *ya* $\dot{5}5$ *ru* 0
> *ya v* 0 *ya* 0 *ru* $\dot{2}50$
>
> Multiplying by four, and adding a number equal to the square of fifty-five (3025), the roots extracted are
>
> *ya* 2 *ru* $\dot{5}5$
> *ya* 0 *ru* 45
>
> Here also a two-fold value is found as before, 50 and 5. But the second is in this case not to be taken: for it is incongruous. People do not approve a negative absolute number.[1]

[3] Two instances are here given to show, that the twofold value is admissible in some cases only. CRISHN.

[1] The second value being five, its fifth part, one, cannot have three subtracted from it. There is incongruity; to indicate which the author adds expressly, 'the second is in this case not to be taken.'

It should be noted that negatives are used in order to find the two positive solutions, but then one of them is rejected on the grounds that "people do not approve a negative absolute number". What is meant is that, in the context of the problem, the acceptance of the solution 5 leads to a negative number of monkeys.

Leonardo of Pisa (Fibonacci), in the first half of the 13th century, interpreted a negative root as a debt, when it appeared in the solution of a problem related to money. Although we also find, later mathematicians, ignoring or rejecting the negatives (Viete, n.n., Frend, n.n., see also below), nevertheless from the 16th century onwards, the negative numbers seem to have been admitted by most mathematicians. The interest in this period, centers on the different ways of introducing the concept in textbooks.

The negative numbers were usually presented in one of two forms:

--by means of "real world" models,

or

-- by means of a "definition", for example, "quantities less than nothing", or "the negatives are of the same nature as the positives, but they have an opposite sense".

Very often these two appear interwoven, with additional comments which emphasize the "realness" of negative numbers, indicating that doubt and doubters existed.

For example, Saunderson (1741):

Of affirmative and negative quantities in Algebra.

2. Algebraic quantities are of two ſorts, affirmative and negative: an affirmative quantity is a quantity greater than nothing, and is known by this ſign +; a negative quantity is a quantity leſs than nothing, and is known by this ſign —: thus + *a* ſignifies that the quantity *a* is affirmative, and is to be read thus, *plus a*, or more *a*: — *b* ſignifies that the quantity *b* is negative, and muſt be read thus, *minus b*, or leſs *b*.

The poſſibility of any quantitie's being leſs than nothing is to ſome a very great paradox, if not a downright abſurdity; and truly ſo it would be, if we ſhould ſuppoſe it poſſible for a body or ſubſtance to be leſs than nothing. But quantities, whereby the different degrees of qualities are eſtimated, may be eaſily conceived to paſs from affirmation through nothing into negation. Thus a perſon in his fortunes may be ſaid to be worth 2000 pounds, or 1000, or nothing, or — 1000, or — 2000;

in which two laſt caſes he is ſaid to be 1000 or 2000 pounds worſe than nothing: thus a body may be ſaid to have 2 degrees of heat, or one degree, or no degree, or — one degree, or — two degrees: thus a body may be ſaid to have two degrees of motion downwards, or one degree, or no degree, or — one degree, or — two degrees, *&c.* Certain it is, that all contrary quantities do neceſſarily admit of an intermediate ſtate, which alike partakes of both extremes, and is beſt repreſented by a cypher or 0: and if it is proper to ſay, that the degrees on either ſide this common limit are greater than nothing; I do not ſee why it ſhould not be as proper to ſay of the other ſide, that the degrees are leſs than nothing; at leaſt in compariſon to the former. That which moſt perplexes narrow minds in this way of thinking, is, that in common life, moſt quantities loſe their names when they ceaſe to be affirmative, and acquire new ones ſo ſoon as they begin to be negative: thus we call negative goods, debts; negative gain, loſs; negative heat, cold; negative deſcent, aſcent, *&c:* and in this ſenſe indeed, it may not be ſo eaſy to conceive, how a quantity can be leſs than nothing, that is, how a quantity under any particular denomination, can be ſaid to be leſs than nothing, ſo long as it retains that denomination. But the queſtion is, whether, of two contrary quantities under two different names, one quantity under one name may not be ſaid to be leſs than nothing, when compared with the other quantity, though under a different name; whether any degree of cold may not be ſaid to be further from any degree of heat, than is lukewarmth, or no heat at all. Difficulties that ariſe from the impoſition of ſcanty and limited names, upon quantities which in themſelves are actually unlimited, ought to be charged upon thoſe names, and not upon the things themſelves, as I have formerly obſerved upon another occaſion; ſee introduction, art. 11. In Algebra, where quantities are abſtractedly conſidered, without any regard to degrees of magnitude, the names of quantities are as extenſive as the quantities themſelves; ſo that all quantities that differ only in degree one from another, how contrary ſoever they may be one to another, paſs under the ſame name; and affirmative and negative quantities are only diſtinguiſhed by their ſigns, as was obſerved before, and not by their names; the ſame letter repreſenting both: theſe ſigns therefore in Algebra carry the ſame diſtinction along with them as do particles and adjectives ſometimes in common language, as in the words convenient and inconvenient, happy and unhappy, good health and bad health, *&c.*

Or MacLaurin (1748):

> A Decrement may be equal to an Increment, but it has in all Operations a contrary Effect; a Motion downwards may be equal to a Motion upwards, and the Depreſſion of a Star below the Horizon may be equal to the Elevation of a Star above it: But thoſe Poſitions are oppoſite, and the Diſtance of the Stars is greater than if one of them was at the Horizon ſo as to have no Elevation above it, or Depreſſion below it. It is on account of this Contrariety that a Negative Quantity is ſaid to be leſs than Nothing, becauſe it is oppoſite to the Poſitive, and diminiſhes it when joined to it, whereas the Addition of o has no Effect. But a Negative is to be conſidered no leſs as a Real Quantity than the Poſitive.

Or Le Blond (1768):

> 16. Les quantités poſitives & les négatives ſont également réelles; elles ne différent qu'en ce qu'elles ſont oppoſées ou priſes en ſens contraire.
>
> 17. Si l'on ſuppoſe que la meſure d'une quantité quelconque, comme, par exemple, celle d'une ligne droite, doive ſe prendre d'Orient en Occident, ou du Midi au Nord, la quantité qui ira dans un ſens oppoſé, c'eſt-à-dire, d'Occident en Orient, ou du Nord au Midi ſera négative, ſi l'on conſidere la premiere comme poſitive; au lieu qu'elle ſeroit poſitive ſi l'autre avoit été regardée comme négative.

The apparent necessity to explain and to convince stands out. Undoubtedly, since these and similar books were written for students, the explanation of the concept by way of examples had a strong didactic justification. Nevertheless, it is also clear, that there was an element of defense against, and refutation of, the opposition. This is very clear in Simpson (1745):

> And farther, to reafon about oppofite effects, and recur to fenfible objects and popular confiderations, fuch as debtor and creditor, *&c.* in order to demonftrate the principles of a fcience whofe Object is abftract Number, appears to me, not well fuited to the nature of fcience, and to derogate from the dignity of the fubject.

These difficulties did not abate for decades thereafter. In fact, they gave birth to a curious phenomenon, the rejectionists. For example, Frend (1796) rejected negatives entirely (Frend, n.n.). He had many objections. Here is how he discusses the introduction of the concept by means of examples.

> ...The firft error in teaching the principles of algebra is obvious on perufing a few pages only in the firft part of Maclaurin's Algebra. Numbers are there divided into two forts, pofitive and negative; and an attempt is made to explain the nature of negative numbers, by allufions to book-debts and other arts. Now, when a perfon cannot explain the principles of a fcience without reference to metaphor, the probability is, that he has never thought accurately upon the fubject.

There were also arguments against the negatives, when they are introduced not by models, but by "definitions" "quantities less that zero" etc.). For example, Carnot (1803) brings possible "contradictions" that derive from these kind of "definitions" (Contradictions in the use of negative numbers,n.n.).

> Les notions qu'on a données jusqu'ici des quantités négatives isolées, se réduisent à deux ; celle dont nous venons de parler, savoir que ce sont des quantités moindres que zéro, et celle qui consiste à dire, que les quantités négatives sont de même nature que les quantités positives, mais prises dans un sens contraire : d'Alembert détruit l'une et l'autre de ces notions. Il repousse d'abord la première par un argument qui me paroît sans réplique.
>
> Soit, dit-il, cette proportion 1 : — 1 :: — 1 : 1 ; si la notion combattue étoit exacte, c'est-à-dire, si — 1 étoit moindre que 0, à plus forte raison seroit-il moindre que 1 ; donc le second terme de cette proportion seroit moindre que le premier ; donc le quatrième devroit être moindre que le troisième ; c'est-à-dire, que 1 devroit être moindre que — 1 ; donc — 1 seroit tout ensemble moindre et plus grand que 1 ; ce qui est contradictoire.
>
> Passons à la seconde notion, qui consiste à dire que les quantités négatives ne diffèrent des quantités positives qu'en ce qu'elles sont prises dans un sens opposé. Cette idée est ingénieuse, mais elle n'est pas plus juste que la précédente. En effet, si deux quantités, l'une positive, l'autre négative, étoient aussi réelles l'une que l'autre et ne différoient que par leurs positions, pourquoi la racine de l'une seroit-elle une quantité imaginaire tandis que celle de l'autre seroit effective ? Pourquoi $\sqrt{-a}$ ne seroit-elle pas aussi réelle que $\sqrt{+a}$? Conçoit-on une quantité effective dont on ne puisse tirer la racine carrée ? et d'où viendroit le privilége que la première $-a$ auroit de donner son signe au produit $-a \times +a$ de l'une par l'autre ? Cette expression de quantités prises en sens contraires l'une de l'autre, est donc au moins déjà très-vague, et mène à une confusion d'idées inextricable. Mais je vais plus loin ; je démontre que la notion est complètement fausse, et que de son admission résulteroient les plus grandes absurdités.

With hindsight, we can appreciate that the arguments with the rejectionists could not be finally settled until a satisfactory formal definition of the negative numbers was available. In fact, it seems that the "battle" with the rejectionists was one of the causes for the search for such a definition, although the concept of negative numbers was old and well accepted by most of the mathematical world. The rejectionist approach was "negative" and destructive. If they can claim a place in history , it is not for their direct contribution, but for their incitement of better and more creative mathematicians to answer them, and thus to lead to an eventually successful mathematical foundation of the concept of negative numbers.

Peacock attempted (Peacock, n.n.), by means of his distinction between arithmetical and symbolical algebra and also by means of the Principle of Permanence of Equivalent Forms that connected the two algebras, to provide a foundation for the concept of negative numbers. His attempt can be considered as one of the stages in the process of the development of the formal definition. The latter, however, as an historical event, is difficult to trace. We found a source in Hamilton (1837), that can be considered as "an attempt to give a system of axioms or principles for analysis" (Mathews, 1978). (We did not find this source suitable for our purposes, and therefore we built a special worksheet for the presentation of the formal mathematical definition not based on a historical source: The formal entry of negative numbers into mathematics, n.n.)

(ii) The arithmetic of negative numbers

As mentioned previously, we found no trace of negative numbers in Greek mathematics. The "rule of signs", as used by them, is the rule of how to deal with expressions of the form (a-b)x(c-d), in which a,b,c,d stand for positive numbers, a>b and c>d.

Concerning Chinese mathematics in ancient times, we find the following (Mikami, 1974).

> The word *chêng* means *positive* and the word *fu negative*. This was therefore the treatment of positive and negative numbers. Although no explanation is tried in the text as to the way in which the positive and negative numbers or rather quantities were represented, yet Liu Hui states expressly in his commentaries that the positive calculators were of the red colour and the negative black.
>
> Well, the "Nine Sections" explains as to the treatment of the positive and negative quantities thus:
>
> "When the equi-named (or equally signed) quantities are to be subtracted and the different named are to be added (in their absolute values), if a positive quantity has no opponent, make it negative; and if a negative has no opponent, make it positive. When the different named are to be subtracted and the same named are to be added (in absolute values), if a positive quantity has no opponent, make it positive; and if a negative has no opponent, make it negative."
>
> From this we are able to see that the author or the revisors of the "Arithmetic in Nine Sections" had known in those old days of the algebraical addition and subtraction of positive and negative numbers or quantities.

In the early Middle Ages, Brahmegupta (Colebrooke, 1817) gave the rules for the four operations:

ALGORITHM.

31. Rule for addition of affirmative and negative quantities and cipher: § 19. The sum of two affirmative quantities is affirmative; of two negative is negative; of an affirmative and a negative is their difference; or, if they be equal, nought. The sum of cipher and negative is negative; of affirmative and nought is positive; of two ciphers is cipher.

32—33. Rule for subtraction: § 20—21. The less is to be taken from the greater, positive from positive; negative from negative. When the greater, however, is subtracted from the less, the difference is reversed. Negative, taken from cipher, becomes positive; and affirmative, becomes negative. Negative, less cipher, is negative; positive, is positive; cipher, nought. When affirmative is to be subtracted from negative, and negative from affirmative, they must be thrown together.

34. Rule for multiplication: § 22. The product of a negative quantity and an affirmative is negative; of two negative, is positive; of two affirmative, is affirmative. The product of cipher and negative, or of cipher and affirmative, is nought; of two ciphers, is cipher.

35—36. Rule for division: § 23—24. Positive, divided by positive, or negative by negative, is affirmative. Cipher, divided by cipher, is nought. Positive, divided by negative, is negative. Negative, divided by affirmative, is negative. Positive, or negative, divided by cipher, is a fraction with that for denominator:[1] or cipher divided by negative or affirmative.[2]

[36 Concluded.] Rule for involution and evolution: § 24. The square of negative or affirmative is positive; of cipher, is cipher. The root of a square is such as was that from which it was [raised].[3]

After Brahmegupta we found no advance, or even evidence of similar achievement, in western mathematics for a considerable period.
In fact, the next time the operations with negative numbers reappeared, followed or preceded by a discussion or justification, is the 17-18th centuries.
The general characteristic of the treatment in this period, is that the discussion of the operations has nothing to do with the way the concept was introduced, and shows the "uncertainty about the logical basis of the subject" (Boyer, 1968). As an example of this we chose the operation of multiplication, and we developed N. Saunderson ,n.n. and L. Euler, n.n.

The rejectionists, whose main arguments were directed at the way the concept of negatives was introduced, had also much to say about the "ridiculous" operations. For example, Frend (1796):

> You may put a mark before one, which it will obey : it ſubmits to be taken away from another number greater than itſelf, but to attempt to take it away from a number leſs than itſelf is ridiculous. Yet this is attempted by algebraiſts, who talk of a number leſs than nothing, of multiplying a negative number into a negative number and thus producing a poſitive number, of a number being imaginary.
>
> ...This is all jargon, at which common ſenſe recoils ; but, from its having been once adopted, like many other figments, it finds the moſt ſtrenuous ſupporters among thoſe who love to take things upon truſt, and hate the labour of a ſerious thought.

Frend proposed an algebra without negative numbers. This led him into interesting "troubles", such as lack of generality in the solution of equations (discussed in Frend, n.n.).

Peacock's Permanence Principle seems to be the first attempt at an answer, that is, to provide a sound basis for the arithmetic of the negative numbers. (This is discussed in Peacock, n.n., together with didactical and heuristical implications.)

Once the negative numbers were formally defined, then for the first time, their arithmetic can be properly developed.

The Irrationals

The historical review that preceded the creation of learning materials on the development of irrationals, was less comprehensive than that undertaken for the history of negative numbers, since the history of irrationals is better documented. Our work, in this case, consisted of the search for, and selection of, suitable sources, which correspond to the following framework.

- The appearance of irrationality in the context of geometry, as the consequence of the incommensurability of two line segments (The Pythagoreans, i.n.).
- The Greek "legitimization" of incommensurability (Euclid, i.n.).

- The "passage" of irrationals from the domain of geometry to that of arithmetic, in the 16th and 17th centuries (Irrationals in the 16th and 17th century, i.n.).

The (almost) 2000 year jump, is due to the fact that, until decimal fractions appeared in the 16th century, there was almost no advance. Mathematicians then came to be concerned about the nature of irrational numbers, mainly because of the infinite decimal representation. In Wallis (1685) there is, apparently for the first time, a detailed treatment of periodic decimal fractions and the distinction between them and what he called "surd" numbers (algebraic irrationals).

- Once the conflict about irrationals had abated (but not disappeared, again because of the absence of a mathematical definition), a further stage in the history of the topic, although not new, was given a wider treatment. This was rational approximations to irrationals (Saunderson, i.n., Bombelli, i.n.).

- The next stage includes the discussion of rationality or irrationality of some given numbers (like pi and e), and the existence of algebraic and transcendental numbers (Introduction and Commentaries, i.n.).

- The last stage is the formalization of irrational numbers; for example, by means of Dedekind cuts (Dedekind, i.n.).

In both sequences of worksheets (negatives and irrationals), many mathematical and didactical issues not described in this chapter, arise and are discussed. A description of the structure and the characteristics of the materials, is the subject of the next chapter.

Chapter 4

THE LEARNING MATERIALS

In this chapter we introduce the learning materials developed for pre and in-service teacher training, their characteristics and their objectives. The learning materials themselves can be found in Part II of this thesis.

Characteristics

i) Relevance

Pre and in-service teacher training should be relevant both in form and in content (Bruckheimer & Hershkowitz, 1983). In our case, relevance in content relates to the history of topics directly connected to the curriculum which the teacher teaches. Relevance in form will be discussed later.

In the process of describing the following characteristics we shall also describe the process of the development.

ii) Primary sources

The materials were developed, as far as possible, around primary sources. Primary sources enable direct contact with creative mathematicians (Mendelssohn, 1912) and thus give the flavour, for instance, of style, notation and printing, and the feeling of the development of mathematics as a dynamic process.

Primary sources have another advantage, they are not about history or interpretation of history, but history itself. The accuracy of primary sources is not a matter for dispute, which secondary or tertiary sources often are (see discussion in Appendix 3). Even when reference is made to primary sources, they are often "translated" into modern language, adding interpretations, which inevitably distances the reader from the mathematical process as it occured.

But the use of primary sources has certain difficulties. The first is that the primary source is often in a language foreign to the learner, and thus must be translated. This violates, in some sense, our goal of "direct exposure" or "direct contact" with the primary source, since sometimes translation in itself involves interpretation. In order to minimize our "interference", we brought the translation side by side with the original source; thus still giving a direct impression of style and notation. Further, we sometimes deliberately included questions which required reference to the source in its original language.

A second problem is that primary sources are often difficult to understand, due both to style and sometimes also due to the mathematics presented.This is especially true for our target population which is not used to reading mathematics. (Teachers mathematics reading ability will be discussed later on in this thesis.)

These problems suggested to us that the materials should take the following form.

iii) Worksheets for active learning

We had to find a way of achieving "mediation" between the primary source, with its language and style difficulties, and the teachers. By mediation we mean the following: "the essence of mediated interaction is that in the process of mediating information, a transformation occurs that facilitates the transmission of meaning that is not inherent in the raw stimulus" (Feuerstein, 1980).
The necessity for such mediation influenced the choice of source material. We had to keep in mind, both the fact that the primary source must represent some feature in the conceptual development of the subject, and that it must not be too difficult to make mediation an impossible task. As we shall see, there were in fact further restrictions on the choice of the sources, in terms of the learning objectives we wished to achieve, as described in the following pages.
Given a particular primary source satisfying the conditions above, we submitted it to a process of "predigestion". By "predigestion" we mean a process in which we analyzed the style, the mathematics implicitly or explicitly involved, the possible value of the source in didactical discussions, the comparison with current mathematical treatments, and of course, we undertook its free translation into the language

of our target population (Hebrew).
The outcome was a sequence of questions designed to help the teacher to rephrase the text using modern notation, to understand the mathematics involved and to lead towards a discussion of mathematical and didactical aspects of the text. (A detailed analysis of the leading questions is brought later on in this chapter.) By the use of such questions we not only wanted to help the teacher to understand the primary source, but also we wanted to avoid the passive learning often associated with mathematics courses in general, and courses in the history of mathematics in particular. Activity in itself is a major source of learning (Dale, 1975), and is particularly important when the teacher is the learner, since "it would also seem reasonable that the teacher should take an active part in the acquisition of content, because if the content is relevant, he will be using almost every single item, either directly or indirectly, in his teaching." (Bruckheimer & Hershkowitz, 1983).
We also have here an aspect of relevance in form to which we alluded above. It is reasonable to suppose that we may encourage, by our example, the teachers to create by themselves an active learning situation in their own classrooms. (This is in accord with the spirit of all the in-service activities in the Mathematics Group of the Science Teaching Department, which is the context within which this thesis grew and took form.)

Another advantage of using this form of learning with source materials, is that the teacher practices reading mathematics. In order to answer the questions in the worksheets, the teacher has to refer to and reread a mathematical text. We therefore expected that, by the end of the "exposure" to one or more of these sequences, the teacher would be able to answer questions on an unseen text. This is not the same thing as the ability to read a mathematical text, because the questions themselves guide the reader, but may contribute towards this end.

iv) <u>Conceptual history</u>

We decided to choose neither a chronological nor a biographical approach as such, but to create a picture of the conceptual development of a topic (Scriba, 1982). Thus the teachers would have the opportunity to discuss the difficulties that mathematicians had in the past, their views, their different approaches to the same topic, the blind alleys, their mathematical creativity, etc. Into that picture, the biographies (with facts, dates and anecdotes) of some mathematicians enter by the way. Each sequence is accompanied by an "Introduction and Commentaries", which contains instructions for anyone wishing to implement the sequence, also showing how the worksheets fit into the framework of the conceptual mathematical story.

Description

The materials were created as sequences of worksheets, whose common title is "A historical source-work collection for in-service and pre-service mathematics teacher courses". The general form of a worksheet is:

- a brief biographical-chronological introduction in order to set the historical scene,
- a historical source, as far as possible a primary source, brought with a free translation into Hebrew (in the Hebrew version),
- leading questions on the source material and on mathematical consequences thereof.

For each worksheet an extensive solution/discussion/answer sheet was prepared, containing detailed solutions to the questions and further source material and background, as appropriate, to complete the historical and the conceptual information. Very often we chose to "relegate good source material" to the answer sheets, not because we expect the teachers to include them in any form in their solutions (e.g. by finding them in libraries, or by "recreating" their arguments), but because we felt that, after the teachers had tried to understand and work through the source material in the worksheet, the further source material and information would be much more meaningful.

Objectives

The general objectives of the series of worksheets are:

- to improve and enrich teachers' mathematical knowledge (on topics related to the curriculum);

- to enrich teachers' didactics, allowing the opportunity for discussion of the demands of didactics as opposed to the demands of pure mathematics and comparisons with present classroom practice;

- to contribute to the creation of a reasonable image of mathematics and mathematical activity;

- to create an awareness of, and positive attitude towards, the history of topics in the curriculum.

In the following we specify the objectives in relation to the topics covered and illustrate them with references to the worksheets.

Mathematics

1) In particular, to be aware of and to understand the mathematical definition of negative and irrational numbers and, more generally, to understand what a formal mathematical definition and proofs are (The formal entry of negative numbers into mathematics, n.n, and Dedekind, i.n.).

2) To improve teachers' ability to read critically mathematical texts (in all the worksheets).

3) To learn (or review) certain mathematical topics (for example, some algebra with polynomials and Descartes' rule of signs in Descartes, n.n.; the solution to quadratic

equations in Frend, n.n.; the concepts of incommensurability and irrationality in The Pythagoreans, i.n. and in Euclid, i.n.; proofs of the irrationality of certain numbers in The Pythagoreans, i.n.and in Irrationals in the 16th and 17th centuries, i.n.; certain methods of approximation to algebraic irrationals in Bombelli, i.n. and in Saunderson, i.n.).

<u>Didactics</u>

1) To distinguish between an intuitive presentation (as is usual in junior-high school) and a formal mathematical presentation of negative and irrational numbers (discussed in almost all the worksheets and especially when the mathematical definition is introduced);

2) To discuss the Principle of Permanence of Equivalent Forms and its value and use as a heuristic guide, usually used in intuitive generalizations, and to compare it with methods and approaches today, both in terms of its mathematical validity and its didactical desirability (Peacock, n.n.);

3) To allow for comparison and discussion of different approaches and explanations to a given topic, for example, multiplication of negative numbers (Saunderson, n.n; Euler, n.n.);

4) To promote an awareness of possible sources of student difficulties and/or misconceptions, through the discussion of the mathematics in the text; for example, the double meaning of the minus sign (Viete, n.n.), the concept of order (Contradictions in the use of negative numbers, Part A, n.n.), and the idea of infinite decimal representation of an irrational number (Irrationals in the 16th and 17th centuries, i.n.).

<u>Mathematical activity and image of mathematics</u>

1) To promote awareness of some of the ways that mathematics evolves from the concrete to the abstract, from the particular to the general, from intuitive heuristical formulation towards formal-axiomatic presentation and the logical necessity of the latter, wherever possible, as a way of removing unclearness, contradictions and to provide a sound basis for argument (Frend n.n., Peacock n.n., The formal entry of negative numbers into mathematics, n.n., Irrationals in the 16th and 17th century, i.n., Dedekind, i.n.);

2) To illustrate different notations and, indirectly, to promote an awareness of "the enormous importance of a good notation. By relieving the the brain of all unnecessary work, a good notation sets it free to concentrate on more advanced problems, and in effect increases the mental power

of the race" (Whitehead, n.d.). Good notation also presents problems in that "Mathematicians have a habit, which is puzzling to those engaged in tracing out meanings, but is very convenient in practice, of using the same symbol in different though allied senses." (Ib.)(Viete n.n., Descartes, n.n., Bombelli, i.n.);

History

1) To provide the teachers with a little historical literacy: names, dates, bibliography, periods, etc. and awareness of the possible errors that may appear in secondary sources (Summary, n.n.)

2) To promote positive attitudes towards reading mathematical texts related to the topics in the curriculum in general, and of historical materials in particular.

The objectives described above are interwoven in almost every worksheet. It may be that one is stressed more than the other in a particular worksheet, because of the characteristics of the source used.

In general, the history of mathematics and the particular form we chose (primary sources-leading questions), provide a rich environment relevant to the above objectives.

Questions in the worksheets

In the following we describe categories of questions constructed to accompany the primary sources. It should be remembered that the objectives are not expected to be achieved by specific questions, but rather by work on one (or more) complete sequences.

Type I: Elementary Understanding

A first step in a guided reading of a primary source, is to help the reader to understand the notation, names of concepts, etc. For this step, questions were designed appropriate to the language and style of the source.
One sort of question of this type is the "dictionary", in which the notation and/or concepts as they appear in the original source are given, and the current forms (or names) are required. In the cases in which the original seems particularly difficult, the current notation is given and the reader is required to identify its equivalent in the source. For example, in Bombelli, i.n., a "dictionary" question accompanyed the extract, as follows.

Modo di formare il rotto nella estrattione delle Radici quadrate.

Molti modi sono stati scritti da gli altri autori dè l'uso di formare il rotto; l'uno tassando, e accusando l'altro (al mio giudicio) senza alcun proposito, perche tutti mirano ad un fine; E ben vero che l'una è più breue dell'altra, ma basta che tutte suppliscono, e quella ch'è più facile, non è dubbio ch'essa sarà accettata da gli huomini, e sarà posta in uso senza tassare alcuno; perche potria essere, che hoggi io insegnassi una regola, laquale piacerebbe più dell'altre date per il passato, e poi venisse un'altro, e ne trouasse una più vaga, e facile, e così sarebbe all'hora quella accetata, e la mia confutata, perche (come si dice) la esperienza ci è maestra, e l'opra loda l'artefice. Però metterò quella che più à me piace per hora, e sarà in arbitrio de gli huomini pigliare qual vorranno: dunque venendo al fatto dico. Che presuposto, che si voglia il prossimo lato di 13, che sarà 3, e auanzerà 4, il quale si partirà per 6 (doppio del 3 sudetto) ne uiene $\frac{2}{3}$, e questo è il primo rotto, che si hà da giongere al 3, che fà $3\frac{2}{3}$, ch'è il prossimo lato di 13, perche il suo quadrato è $13\frac{4}{9}$, ch'è superfluo $\frac{4}{9}$, ma uolendosi più approssimare, al 6. doppio del 3 se gli aggionga il rotto, cioè li $\frac{2}{3}$, e farà $6\frac{2}{3}$, e per esso partendosi il 4, che auanza dal 9 sino al 13, ne uiene $\frac{3}{5}$, e questo si gionge al 3, che fà $3\frac{3}{5}$, ch'è il lato prossimo di 13, di cui il quadrato è $12\frac{24}{25}$, ch'è più prossimo di $3\frac{2}{3}$, ma uolendo più prossimo, si aggionga il rotto al 6 fa $6\frac{3}{5}$, e con esso si parta pur il 4, ne uiene $\frac{20}{33}$, e questo si aggionga, come si è fatto di sopra al 3 fà $3\frac{20}{33}$, ch'è l'altro numero più prossimo, perche il suo quadrato è $13\frac{4}{1089}$, ch'è troppo $\frac{4}{1089}$, e uolendo piu prossimo, partasi 4 per $6\frac{20}{33}$, ne uiene $\frac{66}{109}$, che gionto con il 3 fà $3\frac{66}{109}$, e questo è piu prossimo del passato, che il suo quadrato è $13\frac{96}{11881}$ . . . e così procedendo si puo approssimare à una cosa insensibile.

Method of Forming Fractions in the Extraction of Roots:

Many methods of forming fractions have been given in the works of other authors; the one attacking and accusing another without due cause (in my opinion) for they are all looking to the same end. It is indeed true that one method may be briefer than another, but it is enough that all are at hand and the one that is the most easy will without doubt be accepted by men and be put in use without casting aspersions on another method. Thus it may happen that today I may teach a rule which may be more acceptable than those given in the past, but if another should be discovered later and if one of them should be found to be more vague and if another should be found to be more easy, this [latter] would then be accepted at once and mine would be discarded; for as the saying goes, experience is our master and the result praises the workman. In short, I shall set forth the method which is the most pleasing to me today and it will rest in men's judgment to appraise what they see: meanwhile I shall continue my discourse going now to the discussion itself.

Let us first assume that if we wish to find the approximate root[1] of 13 that this will be 3 with 4 left over. This remainder should be divided by 6 (double the 3 given above) which gives $\frac{2}{3}$. This is the first fraction which is to be added to the 3, making $3\frac{2}{3}$ which is the approximate root of 13. Since the square of this number is $13\frac{4}{9}$, it is $\frac{4}{9}$ too large, and if one wishes a closer approximation, the 6 which is the double of the 3 should be added to the fraction $\frac{2}{3}$, giving $6\frac{2}{3}$, and this number should be divided into the 4 which is the difference between 13 and 9. The result is $\frac{3}{5}$ which, added to the 3 makes $3\frac{3}{5}$. This is a closer approximation to the root of 13, for its square is $12\frac{24}{25}$, which is closer than that of the $3\frac{2}{3}$. But if I wish a closer approximation, I add this fraction to the 6 making $6\frac{3}{5}$, divide 4 by this, obtaining $\frac{20}{33}$. This should be added to the 3 as was done above, making $3\frac{20}{33}$. This is a closer approximation for its square is $13\frac{4}{1089}$ which is $\frac{4}{1089}$ too large. If I wish a closer approximation, I divide 4 by $6\frac{20}{33}$, obtaining $\frac{66}{109}$, [and] add this to 3, obtaining $3\frac{66}{109}$. This is much closer than before for its square is $13\frac{96}{11881}$, which is $\frac{96}{11881}$ too large. If I wish to continue this even further, I divide 4 by $6\frac{66}{109}$ obtaining $\frac{109}{180}$, [the author now continues the process and then remarks:] and this process may be carried to within an imperceptible difference.

... epro cedendo (come ſi è fatto di ſopra) ſi approſſimarà quanto l'huomo vorrà, e ſe bene ci ſono molte altre regole: queſte nondimeno mi ſono parſe le più facili, però a queſte mi atterrò, le quali hò trouato con fondamento, qual non uoglio reſtare di porlo, benche non ſarà inteſo, ſe non da chi intende l'agguagliare di potenze, e tanti eguali à numeri, del quale tratterò nel ſecondo libro à pieno: Però hora parlo ſolo con quelli.

Pongaſi dunque, che ſi habbia à trouare il lato proſſimo di 13, di cui il più proſſimo quadrato è 9; di cui il lato è 3, però pongo che il lato proſſimo di 13. ſia 3. p. 1 tanto, e il ſuo quadrato è 9. piu 6 tanti p. 1. potenza, il qual'è eguale à 13. che leuato 9. a ciaſcuna delle parti, reſta 4, eguale à 6 tanti più 1 potenza. Molti hanno laſciato andare quella potenza, e ſolo hanno agguagliato 6 tanti à 4, che il tanto valeria $\frac{2}{3}$. hanno fatto, che l'approſſimatione ſi è 3 $\frac{2}{3}$ perche la poſitione fù 3. p. 1. tanto, uiene ad eſſere 3 $\frac{2}{3}$, ma uolendo tenere conto della potenza ancora, valendo il tanto $\frac{2}{3}$, la potenza ualerà $\frac{2}{3}$ di tanto, che aggionto con li 6 tanti di prima: ſi hauerà 6 $\frac{2}{3}$ tanti eguale à 4, che agguagliato il tanto valerà $\frac{3}{5}$, e perche fù poſto 3. p. 1. tanto, ſarà 3 $\frac{3}{5}$, e ualendo il tanto $\frac{3}{5}$, la potenza valerà $\frac{3}{5}$ di tanto, e ſi hauerà 6 $\frac{3}{5}$ di tanto eguale à 4, ſi che ſi uede donde naſcono le regole dette di ſopra.

Let us suppose we are required to find the root of 13. The nearest square is 9, which has root 3. I let the approximate root of 13 be 3 plus 1 tanto. Its square is 9 plus 6 tanti p. 1 power. We set this equal to 13. Subtracting 9 from either side of the equation we are left with 4 equal to 6 tanti plus 1 power.

Many people have neglected the power and merely set 6 tanti equal to 4. The tanto then comes to $\frac{2}{3}$ and the approximate value of the root is $3\frac{2}{3}$ since it has been set equal to 3 p. 1 tanto. However, taking the power into account, if the tanto is equal to $\frac{2}{3}$, the power will be $\frac{2}{3}$ of a tanto, which, added to the 6 tanti, will give us $6\frac{2}{3}$ tanti, which are equal to 4. So the tanto will be equal to $\frac{3}{5}$, and since the approximate root is 3 p. 1 tanto it comes to $3\frac{3}{5}$. But if the tanto is equal to $\frac{3}{5}$, the power will be $\frac{3}{5}$ of a tanto and we obtain $6\frac{3}{5}$ tanti equal to 4....

Complete the following "dictionary".

Bombelli	English translation	Modern notation
p. or *piu*	plus	------
equale	equal	------
I tanto	one quantity (unknown)	------
potenza	power (second, of unknown)	------
3 p. I tanto	------	$3 + x$
9. piu 6 tanti p. I potenza	------	------

A variant form is, for example, the open question (which is the first question) in Viete n.n.

Precept II

To subtract a magnitude from a magnitude

Let there be two magnitudes A and B, and let the former be greater than the latter. It is required to subtract the less from the greater.

... subtraction may be fittingly effected by means of the sign of the disjoining or removal[24] of the less from the greater; and disjoined, they will be A "minus" B,...

Nor will it be done differently if the magnitude which is subtracted is itself conjoined with some magnitude, since the whole and the parts are not to be judged by separate laws; thus, if "B 'plus' D" is to be subtracted from A, the remainder will be "A 'minus' B, 'minus' D," the magnitudes B and D having been subtracted one by one.

But if D is already subtracted from B and "B 'minus' D" is to be subtracted from A, the result will be "A 'minus' B 'plus' D," because in the subtraction of the whole magnitude B that which is subtracted exceeds by the magnitude D what was to have been subtracted. Therefore, it must be made up by the addition of that magnitude D.

The analysts, however, are accustomed to indicate the performance of the removal by means of the symbol —....

But when it is not said which magnitude is greater or less, and yet the subtraction must be made, the sign of the difference is: =, i.e., when the less is undetermined; as, if "A square" and "B plane" are the proposed magnitudes, the difference will be: "A square=B plane," or "B plane=A square."

VIETA'S *ANALYTIC ART*

Precept III

To multiply a magnitude by a magnitude

Let there be two magnitudes A and B. It is required to multiply the one by the other. ... their product will rightly be designated by the word "*in* or "*sub*," as, for example, "A in B," by which it will be signified that the one has been multiplied by the other...

If, however, the magnitudes to be multiplied, or one of them, be of two or more names, nothing different happens in the operation.[27] Since the whole is equal to its parts, therefore also the products under the segments of some magnitude are equal to the product under the whole. And when the positive name[28] (nomen adfirmatum) of a magnitude is multiplied by a name also positive of another magnitude, the product will be positive, and when it is multiplied by a negative name (nomen negatum), the product will be negative.

From which precept it also follows that by the multiplication of negative names by each other a positive product is produced, as when "$A=B$" is multiplied by "$D=G$"...

... since the product of the positive A and the negative G is negative, which means that too much is removed or taken away, inasmuch as A is, inaccurately, brought forward (producta) as a magnitude to be multiplied...

... and since, similarly, the product of the negative B and the positive D is negative, which again means that too much is removed, inasmuch as D is, inaccurately, brought forward as a magnitude to be multiplied... ... Therefore, by way of compensation, when the negative B is multiplied by the negative G, the product is positive.

1. Translate all the formulae of *Precept II* and *Precept III* into modern notation.

This sort of question is used when the mathematics is "easier".

A further step in the elementary understanding of the source is achieved with questions which require a reformulation of an idea in modern terms. For example, in Euler, n.n.,

31. Hitherto we have considered only positive numbers; and there can be no doubt, but that the products which we have seen arise are positive also: viz. $+a$ by $+b$ must necessarily give $+ab$. But we must separately examine what the multiplication of $+a$ by $-b$, and of $-a$ by $-b$, will produce.

32. Let us begin by multiplying $-a$ by 3 or $+3$. Now, since $-a$ may be considered as a debt, it is evident that if we take that debt three times, it must thus become three times greater, and consequently the required product is $-3a$. So if we multiply $-a$ by $+b$, we shall obtain $-ba$, or, which is the same thing, $-ab$. Hence we conclude, that if a positive quantity be multiplied by a negative quantity, the product will be negative; and it may be laid down as a rule, that $+$ by $+$ makes $+$ or *plus*; and that, on the contrary, $+$ by $-$, or $-$ by $+$, gives $-$, or *minus*.

33. It remains to resolve the case in which $-$ is multiplied by $-$; or, for example, $-a$ by $-b$. It is evident, at first sight, with regard to the letters, that the product will be ab; but it is doubtful whether the sign $+$, or the sign $-$, is to be placed before it; all we know is, that it must be one or the other of these signs. Now, I say that it cannot be the sign $-$: for $-a$ by $+b$ gives $-ab$, and $-a$ by $-b$ cannot produce the same result as $-a$ by $+b$; but must produce a contrary result, that is to say, $+ab$; consequently, we have the following rule: $-$ multiplied by $-$ produces $+$, that is, the same as $+$ multiplied by $+$ *.

Questions

1. The multiplication of (real) numbers is divided into four cases

   $+\cdot+$, $+\cdot-$, $-\cdot+$ and $-\cdot-$.

   How does Euler deal with each of these?

Type II: Development and transfer

Once the notation and formulation have been rewritten into modern form, we can proceed to the next stage, the learner "does" some mathematics.

i) Mathematical exercises in the style of the text, or inspired by it. For example, in Euclid, i.n.,

DEFINITIONS.

1. Those magnitudes are said to be **commensurable** which are measured by the same measure, and those **incommensurable** which cannot have any common measure.

2. Straight lines are **commensurable in square** when the squares on them are measured by the same area, and **incommensurable in square** when the squares on them cannot possibly have any area as a common measure.

3. With these hypotheses, it is proved that there exist straight lines infinite in multitude which are commensurable and incommensurable respectively, some in length only, and others in square also, with an assigned straight line. Let then the assigned straight line be called **rational**, and those straight lines which are commensurable with it, whether in length and in square or in square only, **rational**, but those which are incommensurable with it **irrational**.

2. Give an example of line-segments which are commensurable in square only.

3. Which of the following statements are correct and which not? Give examples (or counter examples) for each statement.

   a) Segments which are commensurable in length are commensurable in square.

   b) Segments which are incommensurable in square are incommensurable in length.

   c) Segments which are commensurable in square are commensurable in length.

   d) Segments which are incommensurable in length are incommensurable in square.

ii) Completion of missing steps (or justification of steps) in a sequence of statements (for example in Frend n.n., p.4), the finding of hidden assumptions in mathematical arguments (for example in Saunderson, n.n.), and the completion of omissions in the source, which we deliberately made, for example, in Frend, n.n. (p.5),

.... as to the *negative roots* of an equation, they are in truth the real and positive roots of another equation consisting of the same terms as the first equation, but with different signs + and − prefixed to some of them; so that, when writers of Algebra talk of the negative roots of an equation, they, in fact, jumble two different equations together, and suppose the proposed, or first, equation to have not only its own proper roots (which they call its *affirmative*, or *positive*, roots,) but to have likewise the roots of a different equation, which they call its *negative* roots. Thus, for example, they would say, that the quadratick equation $xx + 4x = 320$, has two roots, to wit, the positive, or affirmative, root, , and the negative root, . But this latter number, , is, in truth, the root of a different equation, to wit, of the equation . So that this kind of absurd and fantastick language only tends to the confounding together the two different equations $xx + 4x = 320$, and , and considering them as if they were one and the same equation.

5. a) Fill in the bits we have omitted in the text.

In some cases, a missing proof of a mathematical statement is required (for example, in Descartes, n.n.). These can be relatively difficult questions, and they are designed to encourage the teachers to try to construct proofs, while the complete and detailed answer will be discussed, after their trials, in verbal interaction and finally given in the answer sheet. (This and other aspects of the implementation of the learning materials will be described later.)

Type III: Criticism, analysis and synthesis

Having "done" some mathematics we can move to the next stage, which is to talk about mathematics and relevant didactics.

i) Comparisons between different approaches within the text, or in other texts, or with modern approaches to a topic. For example, in Euler, n.n.,

"What is the difference between Euler's arguments and Saunderson's"? (concerning the presentation of the multiplication of negative numbers).

In this category we also included the analysis and discussion of the mathematical arguments in the text, with the insight and knowledge we have today (as a mathematical historical exercise, taking care not to fall into historically unjustified criticism or judgment upon the level of the mathematics compared to ours, etc). This category relates mainly to the way in which justifications, or proofs of assertions, are mentioned in the historical text. For example, in Bombelli, i.n.,

6. Bombelli writes *"... e cosi procedendo si puo approssimare a una cosa insensibile"* (... and this process may be carried to within an imperceptible difference), but in the extract, he "omits" the fundamental mathematical question, whether, if we continue the process indefinitely, one can prove that the sequence of approximations converges to the desired root. Do it in two stages:

a) Show that the odd approximations are steadily decreasing and the even ones are increasing.
(Remember also what you discovered in the previous question.)

b) Show that the odd and the even approximations converge to the desired root.

ii) Argument and refutation. For example, in Contradictions in the use of negative numbers, n.n.,

....je ne comprens pas que le quarré de — 5 puisse être la même chose que le quarré de + 5, et que l'un et l'autre soit + 25. Je ne sçai de plus comment ajuster cela au fondement de la multiplication, qui est que l'unité doit être à l'une des grandeurs que l'on multiplie, comme l'autre est au produit. Ce qui est également vrai dans les entiers et dans les fractions. Car 1 est à 3, comme 4 est à 12. Et 1 est à 1/3, comme 1/4 est à 1/12. Mais je ne puis ajuster cela aux multiplications de deux moins. Car dira-t-on que + 1 est à — 4, comme — 5 est à + 20 ? Je ne le vois pas. Car + 1 est plus que — 4. Et au contraire — 5 est moins que + 20. Au lieu que dans toutes les autres proportions, si le premier terme est plus grand que le second, le troisiéme doit être plus grand que le quatriéme.

3. What is the "contradiction" that he sees in the use of negative numbers?

4. How would you answer him?

iii) Summary and synthesis. For example, in Viete, n.n., "According to this passage how does Viete regard negative numbers?".

We also include in this category the final exercise in Summary, n.n., in which a suggested division into stages of the history of negative numbers is required.

Chapter 5

IMPLEMENTATION: DESCRIPTION AND ANALYSIS

In this chapter we describe the various ways in which the learning materials on the history of mathematics were implemented, and the analysis of some of these implementation experiences.

DESCRIPTION

The following table summarizes the ways in which we used the materials, illustrating the various implementation possibilities.

	In-service	Pre-service
Frontal	-Two workshops on one sequence each (15 hours). -One workshop on the two sequences ([illegible] hours) -Single worksheets within workshops devoted to other topics	-University course on the two sequences (one semester)
Non frontal	-A correspondence course (10 to 12 months, on one or two sequences)	-Assignments given in parallel with a didactics course (one semester, one sequence)

Frontal meetings

By frontal meetings we do not mean the classical lecture course in which the participants listen passively to a lecturer. In our context, frontal means that the work was done in a classroom environment as a workshop, with one or more tutors, but without formal lecturing. The form of the workshop was as follows. After a brief introduction teachers were given a worksheet and worked in groups or individually, as they pleased. The tutors, including the author of this thesis, "interfered" with supplementary questions to individuals or groups according to their progress - or lack of it. They also responded to questions. Generally, after each worksheet, a collective guided discussion of the questions and their solution took place, with appropriate sign-posting of whence we had come and where we intended to go.
After the discussion, the prepared solution/answer sheet was distributed, both for the further information it contained and also to provide the teachers with a complete record of the activity. Thus the worksheets together with the answer sheets can be seen as replacing the lecture notes. The sequence of worksheets, the verbal discussions and summaries and the answer sheets, together form a conceptual history of the subject.

A less systematic use of the materials was as part of an in-service course (spread over a year) in which the formal mathematics of the negative and rational numbers was the central issue. This course was itself also based on worksheets, answer sheets and collective discussions. Some of the worksheets on the history of the topics were chosen by the tutors, in order to motivate the formal presentations. This illustrates another advantage of developing the historical materials as a sequence of separate worksheets, i.e. it is possible to choose single worksheets and integrate them into other courses.

Non-frontal implementation

By non-frontal ways of implementation we mean that most, or all, the work does not take place in a workshop-classroom environment, but it is supervised or controlled by a tutor, who is responsible for giving the participants appropriate feedback and sign-posting. Thus, for example, the sequences were used as assignments by a lecturer giving a didactics course in a teacher college. Students (prospective teachers) were given the worksheets, one at a time, and were asked to return the written solutions to the lecturer. Then they received the corresponding solution/answer sheet together with the next worksheet. After several worksheets the lecturer devoted a lesson period to a discussion and summary of the work done, as well as to connect and relate topics in the worksheets to the main program of the course.

The other non-frontal way of implementation, was as an in-service correspondence course. Teachers received a worksheet by mail, worked through it and returned their written solutions. A team of three (including the author of this thesis) reviewed the teachers' answers, wrote observations and comments, and returned them together with the corresponding answer sheet and the next worksheet. A summary letter was included to give sign-posting, similar to the verbal summaries in the frontal meetings. Teachers were encouraged to contact the members of the team if they felt that it was necessary.

Both non-frontal ways of implementation have the following advantages. The demand for written solutions may encourage precision in verbal expression, and also written verbalization may improve the ability to recall and organize information (Geeslin, 1977). The response time is considerably greater than in a workshop situation, and the participant is free to choose his own timing and rate of progress, within a reasonable deadline previously stipulated. On the other hand, for some participants, the lack of classroom interaction, or the possibility of working in groups, and joining a collective discussion immediately after a problem arises, can be a disadvantage.

ANALYSIS

The analysis of the implementation experiences was carried out, both in order to correct possible flaws in the materials as a a contribution to their subsequent improvement, and also to obtain mainly qualitative, but also some quantitative, feedback as to whether and to what extent some of the objectives were achieved.

Since our main concern, in this thesis, is qualitative feedback, the observation of individuals is of interest in many cases. A further reason for this is that we are dealing with teachers. If an individual teacher can "benefit" from the materials in any sense (mathematical, didactical or attitudinal), then the result will be worthwhile, considering the number of students that he/she teaches and upon which he/she has influence. We are aware that the information from individuals would be more significant if it were followed by systematic observation of teacher behavior in the classroom, before and after their studying the materials. But this aspect is necessarily beyond the scope of this thesis, since it requires a longitudinal study over an extended period of time.

The analysis to be described was undertaken in the light of our stated objectives and the categories of questions in the worksheets (see Chapter 4). We shall also refer to the different groups of teachers on whom the materials were implemented. In the first place, we shall describe (prospective and in-service) teachers' opinions on the materials and their attitudes towards what they considered they had learned from them.

The teacher's view

According to our general objectives, as listed in the previous chapter, we asked the following question of all the teachers that completed one or two sequences.
"What did you learn in the workshop (course) from the point of view of (a) history, (b) didactics and (c) mathematics?".
This question was answered extensively by almost all the participants.
(a) History. As we saw in chapter 2, most of the teachers in our target population, lacked knowledge of the history of mathematics but were very interested to learn. The responses can be classified in two categories: those who enumerated names of mathematicians or the history of the topic (in general, or quoting specific details), and those who stressed that they became aware of the evolving and dynamic nature of mathematics, as the main contribution of the materials. The following are some illustrative quotations.

> "The different stages of development of the concept of irrationality"

> "The order of the development in my mind was different from that in history".

> "I knew about the development from the naturals to the whole numbers, but I never imagined that there had been arguments and battles, even when the necessity for extension was recognized."

> "A fascinating illustration of the fact that, in many cases the use of concepts far preceded the possibility of human thought to define them in a correct way".

> "The necessity for and the successive attempts to enlarge the different sets of numbers."

(b) Didactics. Most of the teachers pointed out, in various ways, that

> "It became clear that one should distinguish between the didactics of mathematics and pure mathematics"

or,

> "I learned that there are difficulties in explaining mathematical concepts by means of real life models, but it is often difficult to explain them on the basis of abstract definitions. Therefore, it is didactically convenient to integrate the two."

Many people indicated that they learned

> "New ways of explaining things. For example, the multiplication of the negative numbers."

And one teacher added:

> "I received support for the view that no student answer should be dismissed, but one should relate to them all, think about them and discover their rationale."

(c) Mathematics. The most common answer under this heading was related to the formal definitions of negative and irrational numbers. Many teachers indicated that they learned about irrational numbers, in general, or methods of finding rational approximations, in particular.

Of special interest were some comments related to the nature of mathematics, such as,

> "We must 'disconnect' from reality, otherwise the development of mathematics stagnates."
>
> "Ways of defining concepts clearly on the basis of previous concepts."

Another question, which was answered by prospective teachers in the university course and by in-service teachers in the correspondence course (n=13 and n=15 respectively), gives a complementary picture. It illustrates the way teachers regard the objectives of the course.

> It has been proposed to introduce the sequences of worksheets on the negative and irrational numbers as a course in teacher training institutions. Since you are now familiar with the materials, we would like to have your opinion. Therefore, in the following, we list the objectives of the learning materials. You are requested to mark on a 1-10 scale (1 lowest and 10 highest) both:
> - the importance of each objective for the the training of teachers, and
> - the contribution of the sequences to the achievement of that objective.
>
> Objectives
>
> 1) To enhance mathematical knowledge.
>
> 2) To enrich didactical background.
>
> 3) To give training in the reading of mathematical texts.

4) To enhance the awareness of mathematics as a developing subject.

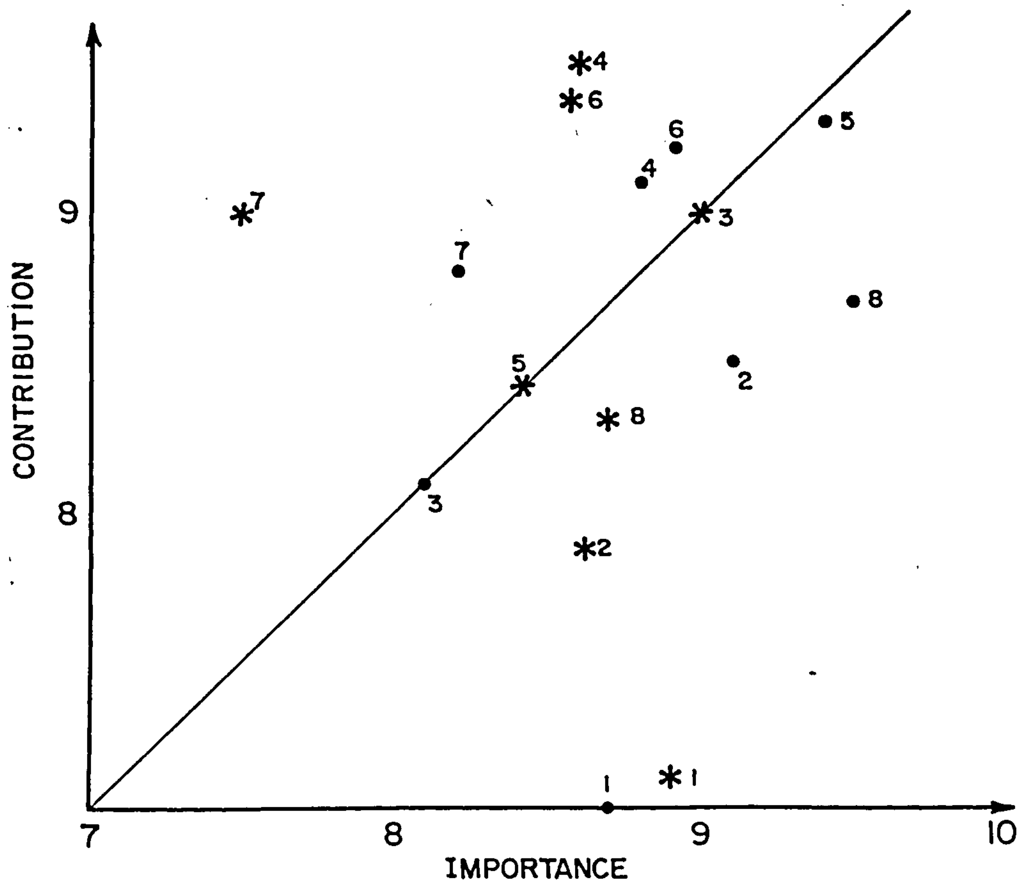
5) To discuss mathematical topics in an interesting way.

6) To learn about the historical development of a topic.

7) To learn about men and their work in the history of mathematics.

8) To enjoy "doing" mathematics.

In the following diagram "contribution" is plotted against "importance". The coordinates of the points indicate the mean obtained for the corresponding question on "importance" and "contribution" respectively. The diagonal bisecting the diagram determines the zones in which the importance attributed by the respondents to a particular objective is respectively greater than, equal to, or less than the contribution to that objective. The two groups, who responded this question, are so small that not too much significance should be attached to the absolute results, nevertheless the patterns are of interest.



* - means obtained from the in-service correspondence course.

• - means obtained from prospective teachers in university course.

At a first glance, we see that almost all the means for "importance" and also for "contribution" are above 8. These high values, and the small differences between importance and contribution (as exhibited by the fact that nearly all the points are close to the diagonal), suggest that, in general, the participants in both groups regarded the course as a serious "contribution" to the achievement of the "important" stated objectives for teacher training.

It is also interesting to note that, the relative positions of the points representing the same statements in both groups with respect to the diagonal, are similar. This suggests that the relative importance of and contribution to the objectives, is independent of teaching experience and may well have some objective validity.
There were objectives to which the materials seemed to "overcontribute" (with respect to the importance attached to them); these were, precisely, the three "historical" objectives, and therefore is not surprising.
The relative position of item 1 would seem to indicate that, teachers' attribution of importance to mathematical knowledge was considerably higher than the contribution they felt the course had given them. This probably means that the mathematical contribution is masked because, although the topic names were familiar, the contents were not.

The overall picture, which emerges from the analysis of the responses to the above questions, indicates clearly that the teachers felt that they had profited from the materials in the sense of the objectives which guided their design.
The following can be considered as indirect evidence supporting the above.
1) In the correspondence course, spread over 10-12 months, 60% of the participants completed the sequence of negative numbers, which is certainly satisfactory for a correspondence course leading to no formal qualification or

professional reward. An examination of those who dropped out showed no pattern related to their background or teaching experience.

2) Both during the work in workshops and courses and in the correspondence course, we witnessed many spontaneous manifestations of enjoyment and satisfaction.

3) The two sequences of worksheets were of interest to many professional academics and teacher trainers abroad. Some of them requested the materials in order to implement them in their own teacher training program.

All the above is important and necessary evidence supporting the introduction of the materials in teacher training. But it is not sufficient. A further step that we undertook in our analysis of the implementation experiences, was to examine the interaction between the teachers and the materials, their difficulties and the real gain that they obtained from the materials.

Teacher difficulties and achievements

This aspect was investigated through the analysis of teachers' responses to questionnaires (see Appendix 2), and through the analysis of the written solutions to worksheets, received in the correspondence course. The data presented below was obtained mainly from the correspondence course and the university course, as well as from the third in-service

workshop.
Here also we shall relate to the objectives as stated in the previous chapter.

Mathematics

An important objective under this heading was to improve mathematics reading ability. For checking the achievement of this objective, two unseen extracts were carefully chosen, according to the following criteria.

- They were taken from primary historical sources, and were thus very similar to those in the worksheets.

- Both were on the same topic: irrationals. The first is a paragraph from MacLaurin's A Treatise of Algebra (1748), and the second is from Chrystal's Algebra, an Elementary Textbook (1886). We chose the first paragraph for the pre-test and the second for the post-test. The first deals with a definition of irrationality, which is the subject of the one of the worksheets, and the second deals with a definition of surd numbers which was not included in the worksheets, and which is not in common use today. In assigning the paragraphs to pre and post-test respectively, we consulted expert opinion. This indicated that the first paragraph, with the questions that we asked thereon, is "easier" than the second. This gives greater validity to the conclusions that we drew. The questions relate to identification of numbers corresponding to a definition or a property given in the paragraph, or require the respondent to indicate whether a given statement related to the content is true or false. According to our classification in the previous chapter, these are transfer and development questions (type II).

In the pre-test, 4 participants (out of 18) did not answer at all, while in the post-test only one person refrained from answering. In the pre-test, no one answered all the items correctly and on the average each person (that answered this question) had 3.5 errors. In the post-test three persons answered all the questions correctly and the mean number of errors per person was 2.4. (It should be pointed out that, in addition to the fact that the paragraph in the post-test was regarded as more difficult, there were also more items to answer than in the pre-test paragraph.)

The improvement, although not dramatic, shows that at least with this type of question, the work with the materials does help towards a better reading of mathematics (in addition to the teachers feeling that it does).

The question of reading ability, although an important objective in this work, was not the sole objective. To understand it fully and to attempt a systematic research would require a complete study of its own. Nevertheless, we can get some further interesting information about the complicated nature of this objective, by the analysis of teachers' responses to questions that we received in the correspondence course.

For example, we looked at one of the questions in Viete n.n., which is at a higher level (type III: criticism, analysis and synthesis). There, the participants were required, as in all the worksheets, to read and understand a mathematical paragraph. In the responses we detected a misconception (or misunderstanding) of the sort that can be rarely detected in a classroom interactive environment. (It should be pointed out that Viete n.n. comes very early on in the course.)

In the extract brought in this worksheet, Viete deals with the rules of operation with "algebraic" expressions. The question we asked was

"According to this passage, how does Viete regard negative numbers?".

We expected the teachers to reread the text, to find the sentences in which the word "negative" is used, and also those corresponding, for example, to the limitations imposed on the operation of subtraction, etc. and to conclude that negative numbers, as we conceive them today, do not appear in the text. Thus we expected them to realise that Viete uses the expression "negative" to mean a quantity that is to be subtracted from another quantity greater than it. The review of the responses to this question showed that about 70% of the respondents (n=55) did not understand the meaning of "negative" as it appears in the paragraph, but regarded it as referring to negative numbers in the modern sense. They completely ignored the clues in the text that indicate that Viete did not allow negative numbers. The use of a word, which belongs to the reader's vocabulary, dominated and suppressed the need for checking its true meaning.

We analyzed the responses to this question in relation to teacher background. The results are given in the following table.

	Correct answer	Incorrect answer	
University graduates	10	7	17
Non University graduates	7	31	38
	17	38	55

Apparently, university graduates have greater experience in reading mathematics and/or are more aware of the fact that the same terminology can be used to indicate different concepts. This may explain the relatively greater success in comparison to non-university graduates.

The relative failure of the respondents in this case, would suggest that, it is desirable, at least in the early stages of such a course, to provide some hints, especially as here, for type III questions. For example, we could reformulate the above question as follows:

"Viete did not admit negative numbers.

a) Find support for this statement in the text.

b) What does he mean when he uses the word "negative"?."

This question looses, perhaps, the investigative aspect of the previous, but avoids misconceptions by carefully guiding the reading.

A similar question was asked in the next worksheet, Descartes n.n.,

"Compare Descartes' and Viète's attitude to negative numbers."

It is interesting to note that, most of the teachers that answered it, succeeded. They applied what they had learned from the Viete answer sheet and from our comments on their answers, and thus made correct comparisons and distinctions.

Another objective under the heading mathematics, was more directly related to content. For example, to learn or review irrational numbers. As we saw in our exploratory study (chapter 2), many teachers did not successfully recognize irrational numbers. A similar picture emerged from the pre-test administered to the in-service and prospective teachers who worked with the materials. They were asked to answer a question in which they had to identify the irrational numbers, among seven given numbers.

In the in-service course pre-test, only two out of 9, identified correctly all the irrationals and in the university course pre-test, 7 out of 12. The principal error in both groups was to indicate 22/7 as irrational.

In the in-service course post-test 7 (of the 9) had no errors in this question (but one person still indicated 22/7 as irrational) and in the university course post-test 10 (out of 12), the other two had a single error, 22/7.

Although, it seems reasonable to conclude that the materials contribute to the recognition of irrational numbers, it is still surprising that, at the end, there are some that confuse a particular irrational number (pi) with one of its rational approximations (22/7). We can speculate whether this is an indication of a general confusion between an irrational number and its rational approximation or, something particular connected with 22/7 and pi. Thus, for example, if in Bombelli, i.n., we obtained 3.6060... as an approximation to root 13 (see answer sheet, p.3), did the student regard this value as an irrational number?. This opens an interesting question that is worth investigating by psychologists of mathematics education.

Didactics

In Chapter 2, we saw teachers' difficulties concerning the presentation of $a^0 = 1$. A similar picture was obtained in the pre-test questionnaires administered in courses and workshops.

Therefore one of the objectives under this heading was to discuss didactical implications of the Principle of Permanence of Equivalent Forms, implicitly used in the classroom to generalize concepts. We wished to compare and contrast its mathematical validity and didactical desirability. This is the subject of Peacock n.n. Our interest centered on the way teachers dealt with the following question (Qu. 3) from the worksheet, because of its immediate relevance to their classroom practice.

3. In view of Peacock's definition of arithmetical algebra, symbolical algebra and the *Principle of Permanence of Equivalent Forms,* find where the principle is used in each of the following and if this use is justified.

   i) We know that

   $$a^m \cdot a^n = a^{m+n}, \qquad m, n \text{ natural numbers.}$$

   Substitute $n = 0$ to obtain

   $$a^m \cdot a^0 = a^{m+0} = a^m,$$

   Whence $a^m \cdot a^0 = a^m$.

   Therefore $a^0 = 1$.

   ii) We know that

   $$(a-b) \cdot (c-d) = a \cdot c - b \cdot c - a \cdot d + b \cdot d,$$

   when $c > d > 0$, $a > b > 0$.

   Substitute $a = 0$, $c = 0$, to obtain

   $$(-b) \cdot (-d) = +b \cdot d .$$

   iii) If a and b represent natural numbers, the distance of $a + b$ from 0 on the number line, is equal to the distance of a from 0 plus the distance of b from 0. We conclude that for every a, b the distance of $a + b$ from 0 is equal to the distance of a from 0 plus the distance of b from 0.

   iv) If a, b and c are natural numbers and

   $$a > b$$

   then $ac > bc$.

   We conclude that for every a, b and c

   $$a > b \implies ac > bc .$$

v) For a, b natural numbers and $a > b$

$$(a^2 - b^2)/(a - b) = a + b \,.$$

We conclude that for every a and b

$$(a^2 - b^2)/(a - b) = a + b \,.$$

vi) For a, m natural numbers

$$a^m = \underbrace{a \cdot a \ldots \cdot a}_{m \text{ times}}$$

Substitute $a = 0$,

whence $0^m = 0$.

Substitute $m = 0$,

whence $0^0 = 0$.

We also looked at the responses to another question (Qu. 4) which, in the Hebrew version, asked the teachers to discuss the approach to the multiplication of negative numbers in a local Israeli text, in the light of Peacock. (In the English version the Israeli text is replaced by an extract from an SMSG text.)

We analyzed 41 written responses of teachers who participated in the correspondence course. About one third of the responses to Qu. 3 were completely correct, and the respondents showed critical understanding of the use of the principle, in the sense that they understood, that in some cases, it is possible to generalize a property from one set of numbers to another, and in other cases not.

An additional 25% of the participants, had errors in one or two items. The most common error was to see the argument presented in section iii) of question 3 as a justified use of the principle (in other words, they did not realised that, if one of the numbers is positive and the other is negative, the generalization leads to a contradiction, as is shown in the answer sheet). But, since the responses were correct in all the other items, we considered that they understood the principle and its use.

We looked in detail at the difficulties of all the other participants. Two interesting misconceptions were found, in their responses. One of them was to the regard the use of the principle as a legitimate basis for a proof, and then to conclude that the principle was not used in any of the cases, since none of them is a proof. Precisely this point is extensively treated in the answer sheet, in which various authors are quoted, stressing the differences between the use of the principle and a proof, but also stressing the didactical and heuristical value of the former. The other misconception is concerned with the results obtained from the argument. If the result is valid as in i), then the conclusion is that the principle was used, and conversely, if the result is false as in iv), then the principle was not used.

Concerning Qu. 4, half of the respondents did not explain, either because they limited themselves to stating that the approach in the text is connected to Peacock's but without indicating how, or because they refrained from answering. The other half indicated that the approach is similar to that of Peacock's, in the sense that we generalize properties from one set of numbers to another, and based on them, obtain new results or definitions. We found additional interesting comments in some of the responses, that related not only to the similarities, but also to the differences. Thus they pointed out that, in the approach in the text, the laws generalized are clearly stated, whereas Peacock's use of the principle is general and diffuse and could (logically) lead to contradictions as well as to correct results.

The notes and comments sent to the teachers together with the answer sheets had some influence. About one third of those who had difficulties with questions 3 and 4 quoted above (Peacock, n.n.), and who also regarded the presentation of $a^0 = 1$ incorrectly (as a proof) in the pre-test, answered this question correctly in the post-test; stating that $a^0 = 1$ is not a proof, it is a definition motivated by the generalization of previous properties.

The same question was administered at the beginning and at the end of the university course. In the pre-test about half answered the question incorrectly, saying that the presentation is a proof. In the post-test almost all of them answered correctly and extensively with references to Peacock's principle.

History and aspects of mathematics and mathematical activity

One of the objectives under this heading was to illustrate different notations. As described in the previous chapter, this was one of the purposes of type I questions (elementary understanding). These questions caused the reader to learn actively about different notations not only paying attention to them, but also translating them into modern form, guided by the questions.
From our observation in the frontal experiences, and from the analysis of the written responses in the correspondence course, almost all the participants succeeded with the "dictionary" questions, and also with the open-ended questions that required reformulation of ideas, formulae, etc.

Another aspect under this heading was to promote awareness of some of the ways that mathematics evolves from the concrete to the abstract, from the particular to the general, from intuitive heuristical formulation towards formal-axiomatic presentation and the logical necessity of the latter, wherever possible, as a way of removing unclearness, contradictions and to provide a sound basis for argument. This was one of the centrals issues in Frend n.n. The participants were requested to answer Frend's arguments (see Frend n.n., pp.3-4). We expected that Frend's rejection of negative numbers would "provoke" them, and in the process of answering him, they would think about aspects of mathematics as stated in the above objective. Again, in this case we looked at the responses in the correspondence course since we had the written record.

The majority of the participants answered extensively, "attacking" Frend with different arguments. One of these arguments can be summarized as follows. If in real life there are situations that can be described by means of negative numbers, this in itself is a sufficient reason for their inclusion in mathematics. For example,

> "I would answer Frend as follows. He argues that it is impossible for 'less than zero' to exist. But in real life there are plenty of examples of 'less than zero': a debt, temperature, etc."

> "The incorporation of negative numbers responds to the necessity of describing debts, temperatures, altitudes etc. There is no 'art' or 'metaphor' in that, this is a stage in the development of mathematics that enables a wider description of phenomena by means of numbers. After all, this was also the way that positive numbers appeared."

Another frequently found argument was centered on the "fact" that mathematics cannot restrict itself, on the contrary, it has to generalize, otherwise there will be unsolved problems. For example,

> "The goal of mathematics is to generalize rules and to enlarge the number system, thus one of the consequences is negative numbers."

> "Frend, with the restrictions that he imposes, reduces considerably, in my view, the 'mathematical world'".

Another category of response refers to didactical aspects of the arguments, defending the presentation of a topic by means of real world examples.

> "In order to introduce a new topic it is good and convenient to use examples known to the learner... It is clear that the negative numbers are numbers on their own also without examples."

A further category of response was to find some logic in Frend's arguments.

> "Maybe there is some logic in his argument, there must be a possibility to look at the numbers in the abstract sense only, disconnected from concrete examples"

Many of the respondents in this category explicitly mentioned that what is lacking is a definition. We found very interesting comments, such as

> "Maybe that, at the time that Frend wrote his arguments, they were quite justified, since as he himself says, until then there was no real attempt to define the negative numbers and also the proofs for their operations. I would try to define them...".

> "The only way to answer him is to build a scientific system with axioms...".

> "If mathematics is a game that we play with symbols on a piece of paper, we can decide upon the rules, that must be clear and not contradictory. Or as Humpty Dumpty said in Alice in Wonderland, 'when I use a word it means just what I choose it to mean - neither more nor less".

(We included this latter comment in the English version of the answer sheet to Frend, n.n., but this respondent had not seen it.)

Another participant, alluding to the absence of mathematical definitions, said:

> "Frend complains about the fact that those who use negative numbers have 'never thought accurately upon the subject'. But did he think accurately upon the arithmetic of positives?..."

The above indicates that the sequence of worksheets on the negative numbers is a motivational and conceptual preparation for the last stage, which is the introduction of the formal definition of negative numbers. Also, based on the responses to Peacock n.n., it seems that the participants have acquired the relevant background to appreciate the differences between the extension to the negative numbers on the basis of the principle of permanence and a logically more complete presentation.

EPILOGUE

A major part of this doctoral thesis was to create the learning materials (brought in Part II). We also investigated their effect as described in the previous chapter. That investigation suggests that, a systematic use of historical materials both in pre- and in-service teacher training have considerable potential to create, both a favorable climate for the learning of mathematics and a less distorted view of mathematical activity, as well as to improve mathematics teaching.

We regard this thesis as a starting point for further studies in a number of possible directions. We believe that creation and implementation (with its subsequent analysis) of similar historical materials on further topics is desirable. Besides the two sequences which have been developed and tested as described, we also present (in Part II) a further sequence on the history of equations. This is an example of another kind of story, since it is not the story of a concept, but it is the story of algebraic techniques. Students, and also some teachers, often tend to solve equations according to some rote algorithmical procedure, without thinking. One of the purposes of the sequence is to study different approaches to the solution of equations, with many didactical asides, concentrating on higher levels of mathematical thinking, rather than the mere technicalities of the solution. The sequence is not complete, since it deals only with topics relevant to junior high school (see Introduction and Commentaries).

Another suggestion for further studies, is to develop materials on more advanced topics. It is reasonable to conclude that a carefully developed sequence on topics from the calculus, for instance, could contribute much to an understanding of the development of a subject that is often presented in a finished and unmotivated form.

Clearly there is room for further controlled long term investigation, on the effects of the materials on teachers, and their practice in the classroom.

A further extension of the present work would be the creation of materials or classroom activities for students. Obviously this can be only meaningful once the teachers have acquired an appropriate historical background. The objectives for those materials require redefinition (removing the didactical stress, for instance), and the mediation will undoubtedly be a particular problem.

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Appendix 1

Questionnaire to prospective teachers (Population A)
(Relevant to Chapter 2)

1) Indicate your degree of agreement with the following statements, according to the scale strongly disagree (1), disagree (2), tend to disagree (3), tend to agree (4), agree (5) and strongly agree (6).
Add comments in each case.

- Mathematical concepts can be used even when their mathematical definition is not known.

- Mathematical textbooks for junior-high school, in general, give the mathematical definition of every new concept.

- The concept of number is one of the concepts that has not changed through thousands of years.

- Mathematics today is different from mathematics three centuries ago.

2. Indicate the irrationals among the following numbers:

(a) $\sqrt{26}$, (b) -7/3 , (c) 1.010010001..., (d) 22/7 , (e) π
(f) $\sqrt{2}/\sqrt{8}$, (g) $\frac{\sqrt{2}-1}{\sqrt{2}+1}$.

3. A teacher presented the following rule to the class.

$$a^m \times a^n = a^{m+n} ,$$

with the justification that

$$a^m = \underbrace{a \times \ldots \times a}_{m \text{ times}} \quad \text{and} \quad a^n = \underbrace{a \times \ldots \times a}_{n \text{ times}}$$

implies

$$a^m \times a^n = \underbrace{a \times \ldots \times a}_{m+n \text{ times}} = a^{m+n} .$$

He then continued: "If we set m=0,

we obtain

$$a^0 \times a^n = a^{0+n} = a^n .$$

Whence $a^0 \times a^n = a^n$

whence $a^0 = 1$ ".

Upon which one student asked:

"Is that a proof that $a^0 = 1$?"

What would your answer be?.Justify.

4. In the following list of mathematical topics and concepts, indicate whether you have learned them or not. If you have learned them indicate whether you feel that you can explain them to others.

- Proof by reductio ad absurdum.
- Continuous fractions.
- Methods of approximating root 2.
- Dedekind cuts.
- Algebraic numbers.
- Definition of negative numbers as equivalence classes of ordered pairs of natural numbers.

5. Have you studied topics from the history of mathematics in the past?

Yes No

If yes, indicate in which of the following ways:

a course, reading, comments from lecturers in other courses, single lectures, workshops, other.

Appendix 2

Questionnaires administered in workshops and courses.

(Relevant to Chapter 5)

Pre-test questionnaire

1. Indicate the irrationals among the following numbers:

(a) $\sqrt{26}$, (b) -7/3 , (c) 1.010010001..., (d) 22/7 ,

(e) π (f) $\sqrt{2}/\sqrt{8}$, (g) $\dfrac{\sqrt{2}-1}{\sqrt{2}+1}$.

2. A teacher presented the following rule to the class.

$$a^m \times a^n = a^{m+n} ,$$

with the justification that

$$a^m = \underset{m \text{ times}}{a \times \ldots \times a} \quad \text{and} \quad a^n = \underset{n \text{ times}}{a \times \ldots \times a}$$

implies

$$a^m \times a^n = \underset{m+n \text{ times}}{a \times \ldots \times a} = a^{m+n} .$$

He then continued: "If we set m=0, we obtain

$$a^0 \times a^n = a^{0+n} = a^n .$$

Whence $a^0 \times a^n = a^n$

whence $a^0 = 1$ ".

Upon which one student asked:

"Is that a proof that $a^0 = 1$?"

What would your answer be?. Justify.

3. The following extract is taken from

MacLaurin C., A Treatise of Algebra. London, 1748.

§ 92. IF a leſſer Quantity meaſures a greater ſo as to leave no Remainder, as $2a$ meaſures $10a$, being found in it five Times, it is ſaid to be an *aliquot* Part of it, and the greater is ſaid to be a *Multiple* of the leſſer. The leſſer Quantity in this Caſe is the *greateſt common Meaſure* of the two Quantities; for as it meaſures the greater, ſo it alſo meaſures itſelf, and no Quantity can meaſure it that is greater than itſelf.

When a third Quantity meaſures any two propoſed Quantities, as $2a$ meaſures $6a$ and $10a$, it is ſaid to be a *common Meaſure* of theſe Quantities; and if no greater Quantity meaſure them both, it is called their *greateſt common* Meaſure.

Thoſe Quantities are ſaid to be *commenſurable* which have any common Meaſure; but if there can be no Quantity found that meaſures them both, they are ſaid to be *incommenſurable*; and ~~and~~ if any one Quantity be called *Rational*, all others that have any common Meaſure with it, are alſo called Rational: But thoſe that have no common Meaſure with it, are called *Irrational* Quantities.

a) Indicate which of the following is an aliquot part of 18a.

9a , 12a , 3a , 4a , 36a.

b) Indicate which of the following pairs of quantities are commensurable, and write down their greatest common measure.

a	$2a$
$8a$	$8a/3$
a	$\sqrt{2}a$
$\sqrt{2}a$	$\sqrt{8}a$
$2a$	$(\sqrt{2} + 1)a$

c) For each of the following sentences, indicate whether they are true or false, according to the definitions in the text.

- $\sqrt{2}$ can be rational.
- A rational and an irrational quantity are always incommensurable.
- If there is no greatest common measure there is no common measure.
- The word irrational in the extract has the same meaning as today.

<u>Post-test questionnaire</u>

1. Indicate the irrationals among the following numbers:

(a) $\sqrt{26}$, (b) $-7/3$, (c) $1.010010001...$, (d) $22/7$,

(e) π (f) $\sqrt{2}/\sqrt{8}$, (g) $\frac{\sqrt{2}-1}{\sqrt{2}+1}$.

2. A teacher presented the following rule to the class.

$$a^m / a^n = a^{m-n} \quad ,(m>n)$$

with the justification that

$$a^m = \underbrace{a\times\ldots\times a}_{m\ \text{times}} \quad \text{and} \quad a^n = \underbrace{a\times\ldots\times a}_{n\ \text{times}}$$

implies (when $a\neq 0$)

$$a^m / a^n = \underbrace{a^{m-n}}_{m+n\ \text{times}} .$$

He then continued: "If we set m=n, we obtain

$$a^m / a^m = a^{m-n} = a^0 .$$

But $a^m / a^m = 1$,

whence $a^0 = 1$ ".

Upon which one student asked:

"Is that a proof that $a^0 = 1$?"

What would your answer be?. Justify.

3. The following extract is taken from

Chrystal, G. <u>Algebra, an Elementary Textbook</u>. Edinburgh, 1886.

Let p denote any *commensurable number;* that is, either an integer, or a proper or improper vulgar fraction with a finite number of digits in its numerator and denominator; or, what comes to the same thing, let p denote a number which is either a terminating or repeating decimal. Then, if n be any positive integer, $\sqrt[n]{p}$ will not be commensurable unless p be the nth power of a commensurable number;* for if $\sqrt[n]{p}=k$, where k is commensurable, then, by the definition of $\sqrt[n]{p}$, $p=k^n$, that is, p is the nth power of a commensurable number.

If therefore p be not a perfect nth power, $\sqrt[n]{p}$ is incommensurable. For distinction's sake $\sqrt[n]{p}$ is then called a *surd number.* In other words, we define a *surd number as the incommensurable root of a commensurable number.*

a) Indicate which of the following numbers are commensurable according to the text.

0.0143 , $\sqrt{5}$, $2-\sqrt{3}$, $\sqrt[5]{32}$, 7.3
$1.307307...$

b) Indicate which of the following numbers are incommensurable according to the text.

$\sqrt[3]{27/64}$, $\sqrt{\sqrt{2}-1}$, $\sqrt{8}$, $\sqrt[4]{16}$
$\sqrt[7]{3/5}$, $\sqrt[3]{36}$.

c) Indicate the surd numbers according to the text.

$\sqrt{\sqrt{5}}$, $\sqrt[5]{10.07}$, $\sqrt{4/9}$, $\sqrt{\sqrt{16}}$, $\sqrt{\sqrt{2}+1}$, π .

d) For each of the following sentences, indicate whether they are true or false, according to the definitions in the text.

- Commensurable numbers are what we call today rational numbers.
- A surd number is what we call today an irrational.
- Every irrational number (in the modern sense) is surd.

- Every incommensurable number is surd.
- Every surd is incommensurable.

Appendix 3

NEGATIVE NUMBERS IN SECONDARY HISTORICAL SOURCES: A SURVEY

It is quite surprising that, the coverage of the topic of negative numbers in the histories of mathematics, is relatively scant. Even the little that exists is not error free.

This section is devoted to the review of the secondary sources on negative numbers, and to errors found.
Works devoted to the development of the number concept (such as Dantzig, 1947; Crossley, 1980) omit, almost completely, any reference to negative numbers. General histories, such as Boyer (1968), Smith (1958), Dedron & Itard (1974), have at most some paragraphs, with very little information about the period and the problems concerning the final acceptance of the negative numbers into mathematics in the second half of the eighteenth century and the first half of the nineteenth century.
"Historical topics for the mathematics classroom" (NCTM, 1969) also devotes almost no space to negative numbers, in spite of the fact that the topic is central to the school curriculum.
Kline (1972) is somewhat more extensive and includes a summary of the state of the acceptance and use of the negatives in the periods 1500-1700 and 1700-1850 respectively, but this extra attention is not without its faults, as we shall see later.

Cantor (1908) and Tropfke (1980) seem to be the most detailed histories which devote attention to the negative numbers.

Errors

"The student or teacher who consults several references to obtain information about some historical point in the development of mathematics may, at first, be shocked to find definite discrepancies." (Read, 1968). In fact, it is true that "In view of the variety of subjects covered in a history of modern mathematics, it seems almost inevitable to introduce to some degree false impressions into such a history..." (Miller, 1921). It is also true that history "is a function of the times. New documents are discovered or deciphered, or old documents are reread with a more critical or understanding mind. The result must be that history, too, changes;..." (Boyer, 1964).
Nevertheless, one must "strive to be as accurate in historical statements as the available evidence permits" (Boyer, 1964), and the reader of secondary sources must be aware of the potential errors.
In the following we shall indicate specific types of errors (with examples from the history of negative numbers) following the spirit of May's (1975) historiographic vices.
1. Factual historical error may be trivial in origin but merits serious attention. It includes, for example, the troublesome question of attribution of priority.

Klein (1924) claims priority for the Principle of Permanence of Equivalent Forms (Peacock, n.n.) for Hankel (1867):

> This was first claimed as a guiding principle in arithmetic by Hermann Hankel, in his *Theorie der komplexen Zahlsysteme**[1], under the name "Prinzip von der Permanenz der formalen Gesetze"**.
>
> * *Theory of Complex Number Systems.*
> [1] Leipzig 1867.
> ** Principle of the permance of formal laws.

but Hankel himself (ib.) gives credit to Peacock:

> In England, wo man Untersuchungen über die Grundprincipien der Mathematik stets mit Vorliebe gepflegt hat, und wo selbst die bedeutendsten Mathematiker es nicht verschmäht haben, in gelehrten Abhandlungen sich mit ihnen zu beschäftigen, ist als derjenige, welcher die Nothwendigkeit einer formalen Mathematik zuerst mit Entschiedenheit erkannt hat, der von seinen Landsleuten sehr geschätzte Cambridger Gelehrte George Peacock zu nennen. In seinem interessanten Report on Certain Branches of Analysis (in dem III. Report of the British Assoc. f. the Advanc. of Science, London 1834, S. 185), ist das Princip der Permanenz freilich einerseits zu eng, andererseits ohne die nöthige Begründung aufgestellt. Die Werke, in denen er dasselbe weiter ausgeführt hat, die Arithmetical Algebra (Cambridge 1842) und die Symbolical Algebra (ebenda 1845) kenne ich ebensowenig wie die einschlagenden Abhandlungen von Augustus de Morgan „On the Foundation of Algebra" (Cambridge Phil. Transact. T. VII, pt. II, 1841 und III, 1842; VIII, pt. II, 1844 und III, 1847). Ueberhaupt ist mir von der zahlreichen Literatur, welche eine von Peacock ausgehende Cambridger Schule über die von ihnen sogenannte „symbolische Algebra" hervorgebracht hat, nur noch eine kurze Abhandlung von D. F. Gregory „On the Real Nature of Symbolical Algebra" (Trans. Roy. Soc. Edinburgh. Vol. XIV, 1840. S. 208) zugänglich gewesen.

The following is a further example. Bourbaki (1960) in a footnote states:

> *** Nous n'entrons pas ici dans l'histoire de l'emploi des nombres négatifs, qui est du ressort de l'Algèbre. Notons pourtant que Bombelli, en ce même lieu, donne avec une parfaite clarté la définition, purement formelle (telle qu'on pourrait la trouver dans une Algèbre moderne), non seulement des quantités négatives, mais aussi des nombres complexes.

The assertion that a pure, formal definition, such as we can find in a modern algebra book, is given with perfect clarity by Bombelli in the latter part of the sixteenth century, is so categoric that it leaves little room for doubt. But we were led to doubt the statement because the pure mathematical formalism which seems to be needed to make such a definition possible, does not seem to have been born before the nineteenth century. And if he did in fact give a proper mathematical definition for negative numbers, one is led to wonder why it was forgotten for some 300 years, although such events did occur in history. In this case, a possible explanation might be that part of Bombelli's work remained unknown until it was published by Bortolotti in 1929.

In our examination of the Bombelli sources, we found no traces of such a definition, both in the three books of L'Algebra published in 1579, nor in books 4 and 5, published in 1929, to which Bourbaki refers.

In order to clarify this question, we wrote to Dr. S.A. Jayawardene, an authority on Bombelli, and his response puts Bourbaki's claim into serious doubt.

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KEW
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Dear Professor Bruckheimer,

Please excuse the delay in replying your letter. It is many years since I ~~last~~ last looked at Bombelli's <u>Algebra</u> and so I had to refresh my memory. Hence the delay.

Bombelli does not <u>define</u> negative numbers (or quantities). He has <u>some</u> idea of negative quantities – he tells us how to operate with them (p.18: "per via del meno"; p.70: "Moltiplicare di più e meno"; "meno 5 via meno 6, fa più 30", etc.; p.73: "meno 28 cavato di ~~meno~~ m. 20 resta ~~più~~ p. 8" etc.). However, there is some ~~embarrassment~~ embarrassment when negative quantities appear in the solutions to equations (pp. 345–346: "la qual valuta è falsa, però tale essempio non si può agguagliare!" The equations are: $x^3+27x+37=9x^2$; or $x^3+18x+25=6x^2$). These are the only equations I can find in Bombelli where the solutions given are negative).

2. Implicit misrepresentation is another source of error in historical writing. It is concerned with the attribution to an author of a particular view without further qualification, when for example,

(i) the author expressed other views on the same point elsewhere,

or

(ii) considerably more significance is attributed to him or to his work, than he (or it) deserves in historical perspective.

Example of (i). In Miller (1933) we find:

> One of the most striking evidences of the fact that the conception of negative numbers was not fully completed before about the close of the eighteenth century is the view that these numbers are greater than infinity which was expressed even in the second half of the eighteenth century by L. Euler, who was one of the greatest mathematicians of all times and corresponded with leading mathematicians of his day.

This statement on Euler appears also in Gardner (1977), in Kline (1972) and in others, without mentioning any source. In our readings of Euler we found no such statement. What we did find (Euler, 1790), was the following which would seem to state exactly the opposite. (If nowhere Euler said it, then this corresponds to category 1 as factual error. Our interpretation to the above assertions, is that somewhere -which we did not find- he did say it, in which case it is an error of type 2 (i).)

§. 100.

Auf eben die Art trifft man in den Reihen öfters unendlich große Glieder an, z. B. in der harmonischen Reihe, deren allgemeines Glied $\frac{1}{x}$ ist, wo zu dem Anzeiger $x = 0$ das unendlich große Glied $\frac{1}{0}$ gehört, und die ganze Reihe folgende ist:

$$\text{ꝛc.} - \frac{1}{4} - \frac{1}{3} - \frac{1}{2} - \frac{1}{1} + \frac{1}{0} + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{ꝛc.}$$

Geht man daher von der Rechten gegen die Linke zu, so wachsen die Glieder, so daß $\frac{1}{0}$ unendlich groß ist, aber so bald die Glieder über diese Grenze kommen, so werden sie negativ und nehmen ab. Hiernach kann man also die unendlich große Größe als eine Grenze betrachten, jenseits welcher die positiven Zahlen negativ werden, und umgekehrt. Dies hat einige verleitet, zu behaupten, daß die negativen Zahlen als Zahlen, die größer seyn als das Unendliche, betrachtet werden könnten, weil die beständig wachsenden Glieder dieser Reihe, nachdem sie das Unendliche erreicht haben, negativ werden. Aber wenn man die Reihe nimmt, deren allgemeines Glied $\frac{1}{xx}$ ist, so werden die Glieder derselben nach dem Uebergange durchs Unendliche wieder positiv

$$\text{ꝛc.} \frac{1}{9} + \frac{1}{4} + \frac{1}{1} + \frac{1}{0} + \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \text{ꝛc.}$$

und davon wird doch wohl Niemand behaupten wollen, daß sie größer sind als das Unendliche.

§. 101.

Oft macht auch das Unendliche in den Reihen die Grenze zwischen den reellen und imaginären Gliedern, wie in der, deren allgemeines Glied $\frac{1}{\sqrt{x}}$ ist:

$$\text{ꝛc.} + \frac{1}{\sqrt{-3}} + \frac{1}{\sqrt{-2}} + \frac{1}{\sqrt{-1}} + \frac{1}{0} + \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \text{ꝛc.}$$

Indeß folgt hieraus nicht, daß die imaginären Größen größer seyen als das Unendliche, denn aus der vorhin angeführten Reihe

$$\text{ꝛc.} + \sqrt{-3} + \sqrt{-2} + \sqrt{-1} + 0 + \sqrt{1} + \sqrt{2} + \sqrt{3} + \text{ꝛc.}$$

würde eben so gut folgen, daß sie kleiner wären.

As an example of type 2 (ii), we bring the following.
The period of the last rejectionists (end of the 18th and begining of the 19th century) and its relevance in the history of negative numbers, is discussed in Chapter 3. Here we shall only set the scene for the description of the error. Most mathematicians at that time, did not accept the rejectionists' drastic conclusion to throw out the negative numbers, although they agreed that they had no mathematically satisfactory answer to the problems that the topic raises (Frend, n.n., and its corresponding answer sheet).
In this context, the attribution to Maseres (a rejectionist) and to his work more significance than it deserves in historical perspective, also gives, by implication, an inflated role to his arguments. For example, (Kline, 1972):

> Masères, who did write acceptable papers in mathematics and a substantial treatise on the theory of life insurance, published in 1759 his *Dissertation on the Use of the Negative Sign in Algebra*. He shows how to avoid negative numbers (except to indicate the subtraction of a larger quantity from a smaller one), and especially negative roots,

Compare this with De Morgan (1915) on Maseres' same work.

> The *Doctrine of Life Annuities*[9] (4to, 726 pages, 1783) is a strange paradox. Its size, the heavy dissertations on the national debt, and the depth of algebra supposed known, put it out of the question as an elementary work, and it is unfitted for the higher student by its elaborate attempt at elementary character, shown in its rejection of forms derived from chances in favor of *the average,* and its exhibition of the separate values of the years of an annuity, as arithmetical illustrations. It is a climax of unsaleability, unreadability, and inutility. For intrinsic nullity of interest, and dilution of little matter with much ink, I can compare this book to nothing but that of Claude de St. Martin, elsewhere mentioned, or the lectures *On the Nature and Properties of Logarithms,* by James Little,[10] Dublin, 1830, 8vo. (254 heavy pages of many words and few symbols), a wonderful weight of weariness.
>
> The stock of this work on annuities, very little diminished, was given by the author to William Frend, who paid warehouse room for it until about 1835, when he consulted me as to its disposal. As no publisher could be found who would take it as a gift, for any purpose of sale, it was consigned, all but a few copies, to a buyer of waste paper.

Within the same category of error, the converse may also be true: one may unreasonable deny or diminish the relevance of some mathematicians. For example (Cajori, 1919).

> Stewart was a pupil of Simson and C. Maclaurin, and succeeded the latter in the chair at Edinburgh. During the eighteenth century he and Maclaurin were the only prominent mathematicians in Great Britain.

"... one wonders why DeMoivre, Taylor and Stirling, all mentioned by Eves (1964, p.355) as eighteenth century mathematicians, were omitted." (Read, 1968).

3. <u>Interpretation</u> (explicit or implicit) may lead to contradictory statements, for example:

"Descartes called negative numbers _false_ and avoided their use" (NCTM, 1969).

"He <Descartes> avoids using negative quantities, which he called _false_, though he does not forbid their use" (Dedron & Itard, 1974, vol. 2)

"He <Descartes> realized the meaning of negative quantities and use them freely" (Bell, 1945).

If one goes back to the original source, it is possible to understand in what sense each of the above may be true or false. It is true that Descartes called the negative numbers false, but the name alone still may not mean very much (compare current terminology, such as imaginary, which is not meant to be taken literally). It is also true that Descartes does not do arithmetic with negative numbers in his _La Geometrie_ (1954 English version), but he certainly deals with them as roots of equations, in which they seem to have an equal status with "true" (positive) numbers.

Another explanation of the origin of contradictory statements like those mentioned above, is given by Miller (1933), in his summary of the "core" of the conceptual history of the negative numbers. "Some of the contradictory historical statements relating to negative numbers are doubtless due to a failure on part of various writers to distinguish between the correct practical use of these numbers and a satisfactory explanation of the theory upon which the correct usage should be based. The latter appeared long after the former."

Another kind of error of interpretation is somehow related to what May (1975) calls logical attribution: "X knew A. B is logical consequence of A. Therefore, X knew B." If we change "logical consequence of" by "logically related to", we find this error, for example, in the following (Gardner, 1977).

> It is important to realize that this attitude was to a large extent a matter of linguistic preference. Greek mathematicians knew that $(10 - 4)(8 - 2)$ equals $(10 \times 8) - (4 \times 8) - (2 \times 10) + (2 \times 4)$. To recognize such an equality is to accept implicitly what later was called the law of signs: the product of any two numbers with like signs is positive, the product of any two numbers with unlike signs is negative. It was just that the Greeks preferred not to call $-n$ a number. To them it was no more than a symbol for something to be taken away. You can take two apples from 10 apples, but taking 10 apples from two apples struck them as senseless. They knew that $4x + 20 = 4$ gives x a value of -4, but they refused to write such an equation because its solution was "not a number." For the same reason $-\sqrt{n}$ was not recognized as a legitimate square root of n.

The above would seem to lead us to conclude that 1) the Greeks knew about negative numbers but avoided their use, and 2) they implicitly accepted the "rule of signs". We did not find any evidence to support this. On the contrary, we did find evidence to support the opposite: the example 4x+20=4 seems to be taken from Diophantus (Heath's version, 1964).

> Their difference is $x^2 - 4x = x(x - 4)$, and the usual method gives $4x + 20 = 4$, *which is absurd*, because the 4 ought to be some number greater than 20.

Concerning the possible implicit acceptance of the "law of signs", Greek mathematicians would most probably have demonstrated the statement (10-4)x(8-2) by reference to a geometrical diagram, whose verbal description is, for example, that given in the worksheet Viete, n.n., in which the concept of negative is entirely absent. It may well be that the confusion arises from the fact that some mathematicians used the term "negative" for a positive number which is to be subtracted from another number bigger than itself. So, the error in interpretation derives from the meaning and notational closeness between the operation of subtraction and the notation for negative numbers.

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THE WEIZMANN INSTITUTE OF SCIENCE
REHOVOT · ISRAEL
DEPARTMENT OF SCIENCE TEACHING

THE NEGATIVE NUMBERS

An historical source-work collection
for in-service and pre-service
mathematics teacher courses

A. Arcavi

M. Bruckheimer

December 1983

- a brief biographical-chronological introduction in order to set the historical scene,
- an historical source; as far as possible a primary source,
- leading questions on the source material and on mathematical and didactical consequences thereof.

To each worksheet an extensive discussion of the solutions, and points arising from them, is given in the respective *answer sheet*. Thus, the answer sheets contain not only the detailed solutions to the questions, but further source material, background, historical and mathematical information. Both the worksheets and the answer sheets are designed as learning materials.

At the present time, this sequence contains 11 worksheets which are intended to be worked in the following order:

1- Introduction.
2- F. Viète.
3- R. Descartes.
4- Contradictions in the use of negative numbers.
5- N. Saunderson.
6- L. Euler.
7- The last of the opposition - W. Frend.
8- G. Peacock.
9- The formal entry of the negative numbers into mathematics. Part I: Definitions.
10- The formal entry of the negative numbers into mathematics. Part II: Proofs. Part III: And so on...
11- Summary.

We have used the sheets in the following:

1

i- In-service workshops.
ii- Pre-service courses.
iii- Correspondence course.
iv- Single worksheets in other workshops.

In each case the work is guided by a tutor. Student-teachers were given the first worksheet. They worked in groups or individually, with the tutor's "interference" if necessary. Then a collective guided discussion took place, and the answersheet was distributed. And so on.

For use with Israeli teachers, we translated the sources in the worksheets freely into Hebrew and gave both the original with the translation, in order to encourage the student to read both (if he understood the original language, if not he could look at it and get some flavour of the period by the form of the print, its elaboration, etc., or even try to identify key words etc. from the translation.) Also the original should be available in cases of misunderstanding attributable to mistranslation. In the present English version there are some extracts in languages other than English, with the corresponding "authorized" translation (for example, Descartes). There are some extracts for which we did not find an "authorized" English translation, in which case we have brought the original only. For English speaking users of these materials, it may be advisable to translate these latter texts freely into English, alongside the original.

This series is not a text. Therefore there is a need for the tutor using the sheets, to add comments, to be in a position to answer questions, and to have in mind the overall structure of the set.

In the following commentary we give briefly, some of the points we had in mind when writing and using the worksheets.

Commentary on the individual worksheets

1- Introduction

In contrast to other worksheets, the introduction is:

- not a *work*sheet, but is intended to be read,
- not a primary source, but a secondary source.

The object is to summarize the history of negatives, thus providing a simple framework in which to work. It also gives information on some early stages in the development of negative numbers, for which we do not bring worksheets. Note that the sentence concerning Viète is erased, because extracts from Viète are the subject of the following worksheet.

2- F. Viète

This worksheet exemplifies the complete avoidance of negative numbers. One of the points being made here is that although the term "negative" and the law of signs appear in the text, the negative numbers as such, are not recognized. The didactics are stressed, since the formulae are relevant to the junior high school curriculum.

3- R. Descartes

The extract exemplifies the use of negatives arising as roots of equations but still not having the "status" of numbers (*racines fausses, le défault d'une quantité)*
We exploit the opportunity to do a little mathematics concerning the roots of polynomials and to prove or justify intuitive or simply stated results in the text.

4- Contradictions in the use of negative numbers

Two contradictions which arose from the use of negative

1

numbers are presented. The student is expected to deal with them using all the mathematics at his disposal. The answer sheet expands on the solution by bringing also the responses of mathematicians of the time. An important point is the general concept of extending properties from one set to a larger set. This can be considered as a prelude to the worksheet on Peacock.

5-6 N. Saunderson - L. Euler

These two worksheets are examples of didactics in the 18th century, which was characterized by the large number of algebra textbooks, in which the authors dealt freely with negatives, and attempted to explain their use.
The purpose is to compare different presentations of the multiplication of negatives, and to emphasize the "problem" of giving a satisfactory explanation when the negatives themselves are not mathematically defined.

7- The last of the opposition - W. Frend

The diminishing, but still present, opposition to the use of negatives at the begining of the 19th century, is exemplified.
The extracts cover:
-arguments against negatives,
-a way of dealing with quadratic equations avoiding negatives completely.
The answer sheet brings extracts from A.N. Whitehead, A. De Morgan and others who, beside countering the rejectionists' arguments, also make general points on

the nature of mathematics.

8- G. Peacock

Peacock made an attempt at formalization in answer to the rejectionists. A discussion with many examples of the didactical implications of the *Principle of Permanence of Equivalent Forms* is brought.

The answer sheet is enriched by extracts from B. Russell, F. Klein and others.

9-10 The formal entry of the negative numbers into mathematics

The previous worksheets are an illustration of the following assertion: "The history of mathematics is a bountiful source of ... examples showing ways in which previous generations have experimented and discovered the need for formal mathematical structures".(Meserve*)

So the end of the story must be a way of introducing the whole numbers formally. We do it by means of equivalence classes of ordered pairs.

These worksheets are not based on historical sources. They are developed by means of a carefully structured sequence of leading questions. The reason for the change of style is that we have not found a satisfactory source on which to base the formal development, and without this stage the story is incomplete.

After working these two sheets we recommend to our students that they re-read the introductory sheet.

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* Meserve B., The History of Mathematics as a Pedagogical Tool. In Zweng M. (ed), Proceedings of the 4th ICME Birkhauser, 1983, p. 398-400.

11- Summary

The sheet consists of two parts. The first brings an extract from a secondary source which summarizes certain periods in the history of the negative numbers, citing mathematicians, books and arguments that the student has met in his work on the sequence.
The objective is to reinforce details from previous sheets. The opportunity is also taken to point out the problem of obtaining reliable historical information.
The second part consists of a chronological table and a suggested division of the development of negatives into stages. As a final exercise the student is asked to correlate the mathematicians and the stages and to fit both into the chronological table.

Final Comment

We do not see these sheets as final or definitive. As we find new and "better" sources, or in the light of experience in the use of the sheets, we make alterations, corrections, replacements, additions, etc.
Other sequences are also in process of development, both with the intention of improving the development per se, and evaluating their implementation.

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The Negative Numbers

Introduction

The following quote is taken from the book by E.T. Bell, The Development of Mathematics. It summarises a part of the history of the negative numbers.

Professional historians are in substantial agreement on the following details in the development of negatives. Diophantus, in the third century A.D., encountering -4 as the formal solution of a linear equation, rejected it as absurd. In the first third of the seventh century, Brahmagupta is said to have stated the rules of signs in multiplication; he discarded a negative root of a quadratic. The rules of signs became common in India after their restatement by Mahavira in the ninth century. Al-Khowarizmi, of about the same time, made no advance except that he appears to have exhibited a positive and a negative root for a quadratic without explicitly rejecting the negative.

Of the Europeans, Fibonacci in the early thirteenth century rejected negative roots, but took a step forward when he interpreted a negative number in a problem concerning money as a loss instead of a gain. It has been claimed that the Indians did likewise. L. Pacioli (1445?–1514, Tuscan) in the second half of the fifteenth century is credited with a knowledge of the rule of signs on such evidence as $(7 - 4)(4 - 2) = 3 \times 2 = 6$. M. Stifel[1] (1487?–1567, German) a fine algebraist for his time, called negative numbers absurd in the middle of the sixteenth century. Cardan, in his *Ars magna* (1545), stated the rule 'minus times minus gives plus' as an independent proposition; he also is said to have recognized negative numbers as 'existent,' but on evidence which seems doubtful. In fact, he called negatives 'fictitious.'

Bombelli in 1572 showed that he understood the rules of addition in such instances as $m - n$, where m, n are positive integers. Vieta, about the same time,...
Finally J. Hudde (1628–1704, Dutch) in 1659 used a letter to denote a positive or a negative number indifferently. As an historical curiosity, it may be mentioned that T. Harriot (1560–1621, English) was one of the first Europeans to duplicate the feat of the ancient Babylonians in permitting a negative number to function as one member of an equation. But he refused to admit negative roots.

With one exception, all items in the foregoing list may be classified as partial extension by formalism. The extension was incomplete because no free use of negatives was made until the seventeenth century. The extension was formal because it had no basis other than the mechanical application of rules of calculation that were known to produce consistent results when applied to positive numbers, and were assumed to be legitimate in the manipulation of negatives. This unbased assumption was to be elevated in the 1830's to the dignity of a general dogma in the notorious and discredited 'principle of permanence of form.'

By the middle of the seventeenth century, untrammeled use of negatives had given mathematicians a pragmatic demonstration that the rules of common algebra lead to consistent results. But there was no attempt to go any deeper and put a substratum of postulates under the rickety formalism.

The one glimmer of mathematical intelligence in the early history of negatives is the suggestion of Fibonacci that a negative sum of money may be interpreted as a loss. This appears to have been the first step toward the second stage in the evolution of negatives, that of interpreting the results of formalism in terms of something which is accepted as consistent. It marks the beginning of two distinct but complementary philosophies of mathematics: the products of mathematical formalism are to be admitted only if they can be put in correspondence with some already established system accepted as self-consistent; all mathematics is a formalism without meaning beyond that implied by the postulates defining the formalism. For example, if Euclidean geometry is accepted as self-consistent, and if the formal algebraic operations with complex numbers can be interpreted in terms of that geometry, the formalism of complex numbers is admissible. This is according to the first philosophy, which was that instinctively and subconsciously adopted by Fibonacci in his encounter with negatives. The second philosophy is illustrated by the rules of algebra in any modern elementary text, where $a, b, c, \cdots, +, \times, =$ are displayed, and it is postulated that $a = a$, $a + b = b + a$, etc.

Each philosophy has greatly enriched mathematics. The first, seeking interpretations, may be called synthetic; the second, beginning and ending in a formalism within its own postulated universe,[2] may be termed analytic.

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1983

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The Negative Numbers

Worksheet: Francois Viète

Francois Viète (or Franciscus Vieta, in the Latin spelling), a Frenchman born in 1540 and died in 1603. Viète studied law and most of his life worked in the public service, devoting his spare time to mathematics, in which he made his name. He wrote books on algebra and geometry. His important work, *In Artem Analyticem Isagoge (Introduction to the Analytic Art)*, appeared in 1591.

One of the innovations attributed to Viète is the use of letters, not only for variables, but also for coefficients. He thus made it possible to deal with general algebraic forms. Even though the notation in his work is much more complicated than that of today, his contribution to the advance of algebra is highly rated and he has been dubbed the "father of modern algebra".

The following is a quotation from the above book. The text is taken from the English translation brought by Klein (1968)*.

Details on the man and his contribution to mathematics can be found in

* Klein J., *Greek Mathematical Thought and the Origin of Algebra* The M.I.T. Press, 1968.

Precept II

To subtract a magnitude from a magnitude

Let there be two magnitudes *A* and *B*, and let the former be greater than the latter. It is required to subtract the less from the greater. ... subtraction may be fittingly effected by means of the sign of the disjoining or removal[24] of the less from the greater; and disjoined, they will be *A* "minus" *B*,...

Nor will it be done differently if the magnitude which is subtracted is itself conjoined with some magnitude, since the whole and the parts are not to be judged by separate laws; thus, if "*B* 'plus' *D*" is to be subtracted from *A*, the remainder will be "*A* 'minus' *B*, 'minus' *D*," the magnitudes *B* and *D* having been subtracted one by one.

But if *D* is already subtracted from *B* and "*B* 'minus' *D*" is to be subtracted from *A*, the result will be "*A* 'minus' *B* 'plus' *D*," because in the subtraction of the whole magnitude *B* that which is subtracted exceeds by the magnitude *D* what was to have been subtracted. Therefore, it must be made up by the addition of that magnitude *D*.

The analysts, however, are accustomed to indicate the performance of the removal by means of the symbol —....

But when it is not said which magnitude is greater or less, and yet the subtraction must be made, the sign of the difference is: =, i.e., when the less is undetermined; as, if "*A* square" and "*B* plane" are the proposed magnitudes, the difference will be: "*A* square=*B* plane," or "*B* plane=*A* square."

3

Precept III

To multiply a magnitude by a magnitude

Let there be two magnitudes A and B. It is required to multiply the one by the other. ... their product will rightly be designated by the word "*in* or "*sub*," as, for example, "A in B," by which it will be signified that the one has been multiplied by the other...

If, however, the magnitudes to be multiplied, or one of them, be of two or more names, nothing different happens in the operation.[27] Since the whole is equal to its parts, therefore also the products under the segments of some magnitude are equal to the product under the whole. And when the positive name[28] (nomen adfirmatum) of a magnitude is multiplied by a name also positive of another magnitude, the product will be positive, and when it is multiplied by a negative name (nomen negatum), the product will be negative.

From which precept it also follows that by the multiplication of negative names by each other a positive product is produced, as when "$A=B$" is multiplied by "$D=G$"...

... since the product of the positive A and the negative G is negative, which means that too much is removed or taken away, inasmuch as A is, inaccurately, brought forward (producta) as a magnitude to be multiplied...

... and since, similarly, the product of the negative B and the positive D is negative, which again means that too much is removed, inasmuch as D is, inaccurately, brought forward as a magnitude to be multiplied... ... Therefore, by way of compensation, when the negative B is multiplied by the negative G, the product is positive.

Questions

1. Translate all the formulae of *Precept II* and *Precept III* into modern notation.

2. Viète explains some of the formulae. Compare his explanations with those which you (or the textbook you use) give.

3. According to this passage, how does Viète regard the negative numbers.

4. The main mathematical preoccupation of the Greeks was geometric. How would they have demonstrated (or justified) the rules in *Precept III*.

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The Negative Numbers

Answer sheet: Viète

FRANÇOIS VIÈTE (VIETA)

1. Precept II

Viète	Equivalent in modern notation
Let there be two magnitudes A and B, and let the former be greater than the latter	A, B A > B
... and disjoined, they will be A "minus" B ...	A - B
... if "B 'plus' D" is to be subtracted from A, the remainder will be "A 'minus' B, 'minus' D" ...	A -(B + D) = A - B - D
But if D is already subtracted from B and "B 'minus' D" is to be subtracted from A, the result will be "A 'minus' B 'plus' D" ...	A -(B - D) = A - B + D

Viète	Equivalent in modern notation
"A square = B plane" or "B plane = A square"	$\lvert A^2 - B \rvert$

Precept III

Viète	Equivalent in modern notation
"A in B"	AB, A·B or A x B
If, however, the magnitudes to be multiplied, or one of them, be of two or more names,... when the positive name of a magnitude is	$A(B + C)$ or $(A + B)(C + D)$
multiplied by a name also positive of another magnitude, the product will be positive,	$A(B + C) = AB + AC$
and when it is multiplied by a negative name the product will be negative	$A(B - C) = AB - AC$

(Note: from the test in Precept II, it is clear that, in this passage, Viète does not mean -B standing on its own, but rather B - C (or B = C), in which he will call, as later "-C" the "negative C".)

... as when "A = B" is multiplied by "D = G",	$(A - B)(D - G)$
since the product of the positive A and the negative G is negative ...	$A(D - G) = AD - AG$
... the product of the negative B and the positive D is negative ...	$(A - B)D = AD - BD$

(Note: By "in as much as A is, in accurately, brought forward as a magnitude to be multiplied", Viète means that we are required to multiply, not by A, but by A - B which is less than A. Hence in the product A(D - G) "too much is removed".)

Therefore, by way of compensation, when the negative B is multiplied by the negative G, the product is positive

$$(A \underline{- B})(D \underline{- G})$$
$$= AD - BD - AG \underline{+ BG}$$

(Note: Viète's justification, although based on arguments of magnitude, is essentially qualitative and not quantitative: we twice subtracted too much, so we have to add some on, but there is essentially no justification that the "compensation" is exact.)

2. The explanation will depend on the textbook and the level of the students. Thus A -(B + D) = A - B - D can be justified, for example, by movements on the number line, or by regarding the - in front of the brackets as equivalent to multiplication by -1, and then using the distributive law.

   The significance of this question is in the discussion of different approaches.

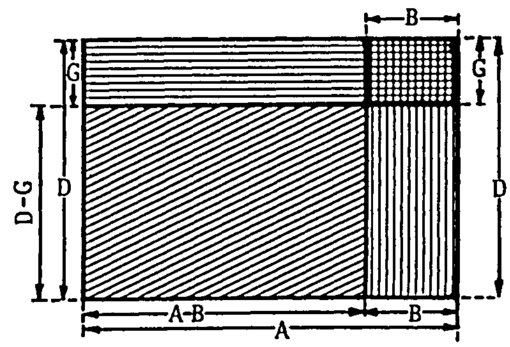
3. This point has been touched upon in the notes written in answer to Question 1, but we will detail it again here. If one reads the quotes from Viète superficially, one may get the impression that he deals with negative numbers and that he even "deduces" the law of signs in Precept III. But Viète did not admit negative numbers in any sense, as can be seen from

i) In order to perform the subtraction "A 'minus' B", the first quantity must be greater than the second.

ii) If one does not know which of the two quantities is the greater, then Viète uses the special symbol =, to indicate that whichever of the two is smaller is to be subtracted, i.e. to avoid obtaining a negative result, which he cannot admit.

iii) The law of signs is brought without his recognising negative numbers. Viète was not the first in this. The Greeks were aware of the extended distributive law (see also the answer to question 4) and that the product of two *subtracted numbers* (as opposed to *negative numbers*) gives a result which has to be added and not subtracted. Thus the use of law of signs does not imply acceptance of the negative numbers. In all our reading of historical source material, we have to be careful not to attach unjustified modern meaning to what we read. The strength of our modern symbolism is its generality, but in deducing that Viète's Precept III can be written in the form $(A - B)(C - D) = AC - AD - BC + BD$, we have to be careful to remember the limitations he imposes, i.e. $A > B$, $C > D$.

4. In answer to this question we do not expect a historical faithful demonstration (see later), but rather something like the following, which is also found in modern school texts.



The area of the rectangle whose sides are of length A - B and D - G is (A - B)(D - G). But, from the figure, we can obtain this area as the area of the large rectangle (A•D), less the area of the two bounding rectangular strips (A•G and B•D), except that we have removed the small rectangle (B•G) twice. So we have to add it back on once.

Since Viète was influenced by Diophantus*, it seems reasonable to suggest that he described in words and "rudimentary" symbolism, what the Greeks drew, so helping in the process of giving algebra an existence separate from geometry.

To get the flavour of an original Greek proof of what we today regard as an algebraic result, we bring Proposition 7 from Book II of *Euclid's Elements***.

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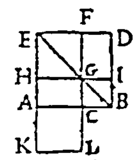
* Diophantus of Alexandria lived in the third century C.E. According to Kline, M., *Mathematical Thought from Ancient to Modern Times,* O.U.P., 1972, "The highest point of Alexandrian Greek algebra is reached with Diophantus." His major extent work is the apparently incomplete *Arithmetica*.

** Taken from the English version (7th edition) of de Chales' Euclid (London, 1726).

PROPOSITION VII.

A PROBLEM.

If a Line be divided, the Square of the whole Line with that of one of its Parts, is equal to two Rectangles contain'd under the whole Line, and that firſt Part, together with the Square of the other Part.



LET the Line AB be divided any where in C; the Square AD of the Line AB, with the Square AL, will be equal to two Rectangles contain'd under AB and AC, with the Square of CB. Make the Square of AB, and having drawn the Diagonal EB, and the Lines CF and HGI; prolong EA ſo far, as that AK may be equal to AC; ſo AL will be the Square of AC, and HK will be equal to AB; for HA is equal to GC, and

and GC, is equal to CB, becauſe CI is the Square of CB (*by the Coroll. of the* 4.)

Demonſtration.

'Tis evident, that the Squares AD and AL are equal to the Rectangles HL and HD, and the Square CI. Now the Rectangle HL is contain'd under HK equal to AB, and KL equal to AC. In like manner the Rectangle HD is contain'd under HI equal to AB, and HE equal to AC. Therefore the Squares of AB and AC are equal to two Rectangles contain'd under AB and AC, and the Square of CB.

In Numbers.

Suppoſe the Line AB to conſiſt of 9 Parts, AC of 4, and CB of 5. The Square of AB 9 is 81, and that of AC 4 is 16; which 81 and 16 added together make 97. Now one Rectangle under AB and AC, or four times 9, make 36, which taken twice, is 72; and the Square of CB 5 is 25; which 72 and 25 added together make alſo 97.

PRO

Algebraically, this result is equivalent to:

$$a^2 + b^2 = 2ab + (a - b)^2$$

where AB = a and AC = b.

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<u>The Negative numbers</u>

<u>Worksheet: R. Descartes</u>

LA
GEOMETRIE
DE
RENÉ DESCARTES

A PARIS,
Chez Charles Angot, ruë ſaint Iacques,
au Lion d'or.

M. DC. LXIV.
AVEC PRIVILEGE DV ROY.

René Descartes (1596-1650) was a Frenchman, famous both as mathematician and philosopher. He received a religious (Christian) education and studied law. For some years he served in the army, and in the course of his duties, travelled to many European countries, where he met with mathematicians and philosophers. His most famous work is the *Discours de la Méthode* which was published in 1637. One of the appendices to this work is called *La Géometrie*. After some years this appendix began to be published as a separate work, as shown, for example, in the above title page of the 1664 edition.

Descartes is credited, among other things, with having made a further significant step in the process of developing analytical geometry, as well as the introduction of new mathematical symbols, some of which have remained in use to the present day.

The following facsimile is taken from the first edition of *La Géometrie* with its corresponding English translation,* as it appears in Smith D. E. & Latham M. L. , *The Geometry of René Descartes,* Dover, 1954.

… il faut que ie die quelque choſe en general de la nature des Equations; c'eſt a dire des ſommes compoſées de pluſieurs termes partie connus, & partie inconnus, dont les vns ſont eſgaux aux autres, ou plutoſt qui conſiderés tous enſemble ſont eſgaux a rien. car ce ſera ſouuent le meilleur de les conſiderer en cete ſorte.

Combien il peut y auoir de racines en chaſq; Equatiõ.

Scachés donc qu'en chaſque Equation, autant que la quantité inconnue a de dimenſions, autant peut il y auoir de diuerſes racines, c'eſt a dire de valeurs de cete quantité. car par exemple ſi on ſuppoſe x eſgale a 2; oubien $x - 2$ eſgal a rien ; & derechef $x \infty 3$; oubien $x - 3 \infty 0$; en multipliant ces deux equatiõns $x - 2 \infty 0$, & $x - 3 \infty 0$, l'vne par l'autre, on aura $xx - 5x + 6 \infty 0$, oubien $xx \infty 5x - 6$, qui eſt vne Equation en laquelle la quantité x vaut 2 & tout enſemble vaut 3. Que ſi derechef on fait $x - 4 \infty 0$, & qu'on multiplie cete ſomme par $xx - 5x + 6 \infty 0$, on aura $x^3 - 9xx + 26x - 24 \infty 0$, qui eſt vne autre Equation en laquelle x ayant trois dimenſions a auſſy trois valeurs, qui ſont 2, 3, & 4.

---

* The footnotes of the translation are ommited here.

5

... some general statements must be made concerning the nature of equations. An equation consists of several terms, some known and some unknown, some of which are together equal to the rest; or rather, all of which taken together are equal to nothing; for this is often the best form to consider.[180]

Every equation can have[181] as many distinct roots (values of the unknown quantity) as the number of dimensions of the unknown quantity in the equation.[182] Suppose, for example, $x = 2$ or $x - 2 = 0$, and again, $x = 3$, or $x - 3 = 0$. Multiplying together the two equations $x - 2 = 0$ and $x - 3 = 0$, we have $x^2 - 5x + 6 = 0$, or $x^2 = 5x - 6$. This is an equation in which x has the value 2 and at the same time[183] x has the value 3. If we next make $x - 4 = 0$ and multiply this by $x^2 - 5x + 6 = 0$, we have $x^3 - 9x^2 + 26x - 24 = 0$ another equation, in which x, having three dimensions, has also three values, namely, 2, 3, and 4.

Quelles sont les fausses racines.

Mais souuent il arriue, que quelques vnes de ces racines sont fausses, ou moindres que rien. comme si on suppose que x designe aussy le defaut d'vne quantité, qui soit 5, on a $x + 5 \infty 0$, qui estant multipliée par $x^3 - 9xx + 26x - 24 \infty 0$ fait

$$x^4 - 4x^3 - 19xx + 106x - 120 \infty 0$$

pour vne equation en laquelle il y a quatre racines, a sçauoir trois vrayes qui sont 2, 3, 4, & vne fausse qui est 5.

Cōment on peut diminuer le nombre des dimensions d'vne Equation lorsqu'on connoist quelqu'vne de ses racines.

Et on voit euidemment de cecy, que la somme d'vne equation, qui contient plusieurs racines, peut tousiours estre diuisée par vn binóme composé de la quantité inconnuë, moins la valeur de l'vne des vrayes racines, laquelle que ce soit; ou plus la valeur de l'vne des fausses. Au moyen de quoy on diminue d'autant ses dimensions.

Cōment on peut examiner si quelque quantité donnée est la valeur d'vne racine.

Et reciproquement que si la somme d'vne equation ne peut estre diuisée par vn binóme composé de la quantité inconnue + ou - quelque autre quantité, cela tesmoigne que cete autre quantité n'est la valeur d'aucune de ses racines. Comme cete derniere

$$x^4 - 4x^3 - 19xx + 106x - 120 \infty 0$$

peut bien estre diuisée, par $x - 2$, & par $x - 3$, & par $x - 4$, & par $x + 5$; mais non point par x + ou - aucune autre quantité. cequi monstre qu'elle ne peut auoir que les quatre racines 2, 3, 4, & 5.

Combien il peut y auoir de vrayes racines en chasque Equatiō.

On connoist aussy de cecy combien il peut y auoir de vrayes racines, & combien de fausses en chasque Equation. A sçauoir il y en peut auoir autant de vrayes, que les signes + & - s'y trouuent de fois estre changés; & autant de fausses qu'il s'y trouue de fois deux signes +, ou deux signes - qui s'entresuiuent. Comme en la derniere, a cause qu'aprés $+ x^4$ il y a $- 4x^3$, qui est vn changement du signe + en -, & aprés $- 19xx$ il y a $+ 106x$, & aprés $+ 106x$ il y a $- 120$ qui sont encore deux autres changemens, on connoist qu'il y a trois vrayes racines; & vne fausse, a cause que les deux signes -, de $4x^3$, & $19xx$, s'entresuiuent.

It often happens, however, that some of the roots are false[187] or less than nothing. Thus, if we suppose x to represent the defect[188] of a quantity 5, we have $x+5=0$ which, multiplied by $x^3-9x^2+26x-24=0$, yields $x^4-4x^3-19x^2+106x-120=0$, an equation having four roots, namely three true roots, 2, 3, and 4, and one false root, 5.[189]

It is evident from the above that the sum[190] of an equation having several roots is always divisible by a binomial consisting of the unknown quantity diminished by the value of one of the true roots, or plus the value of one of the false roots. In this way,[191] the degree of an equation can be lowered.

On the other hand, if the sum of the terms of an equation[192] is not divisible by a binomial consisting of the unknown quantity plus or minus some other quantity, then this latter quantity is not a root of the equation. Thus the[193] above equation $x^4-4x^3-19x^2+106x-120=0$ is divisible by $x-2$, $x-3$, $x-4$ and $x+5$,[194] but is not divisible by x plus or minus any other quantity. Therefore the equation can have only the four roots, 2, 3, 4, and 5.[195] We can determine also the number of true and false roots that any equation can have, as follows:[196] An equation can have as many true roots as it contains changes of sign, from + to — or from — to +; and as many false roots as the number of times two + signs or two — signs are found in succession.

Thus, in the last equation, since $+x^4$ is followed by $-4x^3$, giving a change of sign from + to —, and $-19x^2$ is followed by $+106x$ and $+106x$ by -120, giving two more changes, we know there are three true roots; and since $-4x^3$ is followed by $-19x^2$ there is one false root.

De plus il eſt ayſé de faire en vne meſme Equation, que toutes les racines qui eſtoient fauſſes deuienent vrayes, & par meſme moyen que toutes celles qui eſtoiēt vrayes deuienent fauſſes : a ſçauoir en changeant tous les ſignes + ou -- qui ſont en la ſeconde, en la quatrieſme, en la ſixieſme, ou autres places qui ſe deſignent par les nombres pairs, ſans changer ceux de la premiere, de la troiſieſme, de la cinquieſme & ſemblables qui ſe deſignent par les nombres impairs. Comme ſi au lieu de

Cōment on fait que les fauſſes racines d'vne Equation deuienēt vrayes, & les vrayes fauſſes.

$$+x^4 - 4x^3 - 19xx + 106x - 120 \propto 0$$

on eſcrit

$$+x^4 + 4x^3 - 19xx - 106x - 120 \propto 0$$

on a vne Equation en laquelle il n'y a qu'vne vraye racine, qui eſt 5, & trois fauſſes qui ſont 2, 3, & 4.

5

It is also easy to transform an equation so that all the roots that were false shall become true roots, and all those that were true shall become false. This is done by changing the signs of the second, fourth, sixth, and all even terms, leaving unchanged the signs of the first, third, fifth, and other odd terms. Thus, if instead of

$$+x^4-4x^3-19x^2+106x-120=0$$

we write

$$+x^4+4x^3-19x^2-106x-120=0$$

we get an equation having one true root, 5, and three false roots, 2, 3, and 4.[107]

Questions

1. Complete the "dictionary" in the following table.

Descartes	Today
quantité inconnue	
	degree of an equation
--	
rien	
	=
	x^2
	additive inverse (of a positive number)
racines fausses	
somme d'une équation	

2. What does Descartes have to say about the connection between the number of roots and the degree of an equation?

3. Using the method which Descartes suggests, construct the equation with the two "true" roots 1 and 2, and the two "false" roots 2 and 3.

4. Descartes states a necessary and sufficient condition for a polynomial P(x) to be divisible by the expression x - a.
What is this condition? Prove it!

5. In the equation given in the text Descartes makes an assertion about the number of its positive and negative roots.
   a. Which fact does he use in his argument?
   b. Is his conclusion justified?
      (Consider for example $x^3 + x^2 + x - 3 = 0$)

6. Descartes discusses the connection between the number of positive and negative roots and the number of changes or non-changes of sign in the terms of an equation. The resulting rule has become known as *Descartes' Rule of Signs*.
   a) How would you formulate the rule in the light of your answer to Qu. 5?
   b) How would you justify the rule in general?

7. a) What change does Descartes suggest in a given equation in order to turn positive into negative roots and vice-versa?
   b) Prove that it works.

8. Compare Descartes' and Viète's attitude to negative numbers.

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The Negative Numbers

Answer sheet: R. Descartes

Note → *the mistake in the title of the book.*

1.

Descartes	Today
quantité inconnue	unknown, variable
dimensions de la quantité inconnue	degree of an equation
--	-
rien	zero
ꝏ	=
xx	x^2
le défaut d'une quantité	additive inverse (of a positive number)
racine fausse	negative root
somme d'une équation	polynomial (or function)*

* Descartes says that an equation can be written in two forms; for example,

xx ꝏ 5x -- 6

or xx -- 5x + 6 ꝏ 0 ,

which we would write as

$$x^2 = 5x - 6$$

or $$x^2 - 5x + 6 = 0$$

Following some of his predecessors, Descartes prefers the latter form. The left-hand side of the equation he calls *somme d'une équation*, which we would call a polynomial (or function).

★★★

As mentioned in the worksheet, Descartes has an important place in the development of mathematical symbolism, and we will enlarge a little on one or two aspects*.

- The symbol ꝏ to designate equality. Some explain this symbol as a combination of the letters a and e (æ) and their reverse. These two letters are the first letters of the Latin *aequalis*, meaning "equal". Others maintain that ꝏ is astronomic symbol for *Taurus*, and yet others that it is an arbitrary invention. The = symbol which we use today, apparently first appeared in the book *Whetstone of Witte*, published in London in 1557 by Robert Recorde (1510 - 1558), who wrote

  > I will sette as I doe often in woorke use, a paire of paralleles, or Gemowe lines of one lengthe, thus: =====, bicause noe .2. thynges, can be moare equalle.

  That is, no two things are "more equal" than two parallel lines. The fact that Descartes, nearly a century later, was using ꝏ , shows that it took a long while for the = symbol to become universally established.

- The use of small letters in algebra. This usage remained with us. In particular, the use of the last letters of the alphabet, x, y, z to represent unknowns, and the first letters, a, b, c to denote known quantities, is due to Descartes.

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* Further details on Descartes' symbolism, and on the history of mathematical symbolism in general, can be found in
6 Cajori, F., *A History of Mathematical Notation*, Vol. I, Open Court, 1928.

- The designation of exponents by a small number to the upper right of the letter as in x^3 is also due to Descartes, although he himself occasionally writes (as in the extract) xx instead of x^2.
- The use of -- for subtraction. This symbol is characteristic of a number of works in the first half of the seventeenth century.

2. Descartes writes:

> Scachés donc qu'en chasque Equation, autant que la quantité inconnue a de dimensions, autant peut il y auoir de diuerses racines, c'est a dire de valeurs de cete quantité.

Today we have the *Fundamental Theorem of Algebra* which states that an equation of degree n has n roots, where all roots are taken into account - positive, negative, equal (according to their multiplicity) and complex. Descartes writes *peut il y avoir* and not "must have" or "has", since, apparently, he was referring to real roots and then only to different roots - a multiple root probably being counted only once. Hence the number of roots is equal to or less than the degree of the equation (less, if there are complex or multiple roots).

The fundamental theorem of algebra was, it seems, first enunciated by Girard (1595-1632). Partial attempts at a proof were made by Euler (1707-1783) and D'Alembert (1717-1783). The first complete was given by Gauss (1777-1855) in his doctoral thesis in 1799*.

The proof of the theorem is not easy. But, if we assume that every polynomial equation has at least one root (real or complex), then the remainder becomes easy to prove. For if

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* Further details and sources connected with this theorem can be found in, for example, Struik, D.J., *A Source-Book in Mathematics, 1200-1800,* Harvard U.P., 1969.

a is the root, divide the polynomial by $x - a$, to obtain a new polynomial of degree one less than the original. This new polynomial also has at least one root, by our assumption, and hence the process can be continued until the original polynomial has been completely factorised. Clearly the number of roots is equal to the degree of the equation.

3. $(x - 1)(x - 2) = x^2 - 3x + 2$ and hence $x^2 - 3x + 2 = 0$ is an equation with the two "true" (positive) roots 1 and 2. If we now multiply by $x + 2$ and then by $x + 3$, we shall obtain an equation with four roots: the two "true" roots 1 and 2 and the two "false" (negative) roots 2 and 3.

$$(x^2 - 3x + 2)(x + 2) = x^3 - x^2 - 4x + 4$$
$$(x^3 - x^2 - 4x + 4)(x + 3) = x^4 + 2x^3 - 7x^2 - 8x + 12.$$

4. Descartes maintained that the necessary and sufficient condition for a polynomial $P(x)$ to be divisible by $x - a$, in that a be a root of the equation $P(x)$, i.e. $P(a) = 0$.

Given the polynomial $P(x)$, we have to prove that

$P(x)$ is divisible by $(x - a)$ $\Leftrightarrow$ $P(a) = 0$, that is a root of $P(x) = 0$.

i) We start by proving $\Rightarrow$.
Suppose $P(x)$ is divisible by $x - a$, that is

$$P(x) = (x - a)\, Q(x),$$

where $Q(x)$ is a polynomial of degree one less than $P(x)$. If we substitute x for a, we obtain

$$P(a) = (a - a)\, Q(a) = 0$$

6

ii) Now to prove $\Leftarrow$.

Suppose that P(a) = 0 and that, contrary to what we have to prove, P(x) is not divisible by x - a, that is

$$P(x) = (x - a)\, Q(x) + R,$$

where $R \neq 0$. Substituting x for a, we obtain

$$P(a) = 0 + R \neq 0.$$

But we begin by supposing P(a) = 0. Hence the supposition that P(x) is not divisible by x - a leads us to a contradiction. Thus we conclude that P(a) = 0 $\Rightarrow$ P(x) divisible by (x - a).

5. a. Descartes states with care the following rule:

> . . . y en peut auoir autant de vrayes, que les fignes + & -- s'y trouuent de fois eftre changés ; & autant de fauffes qu'il s'y trouue de fois deux fignes +, ou deux fignes -- qui s'entrefuiuent.

Afterwards, he gives an example and concludes from it

> . . . on connoift qu'il ya . . .

b. Consider the example

$$x^3 + x^2 + x - 3 = 0$$

and following Descartes we conclude that the equation has 2 negative roots (two non-changes of sign) and one positive root (one change of sign). But this is not correct: the equation has one positive root 1, and the other two roots are not real.

6. a) Descartes' conclusion is justified on condition that all the roots are real. Otherwise, the rule has to be stated as follows:

The number of positive roots is less than or equal to the number of changes of sign in successive terms of the polynomial, and the number of negative roots is less than or equal to the number of non-changes of sign. As already mentioned, Descartes referred to real numbers and different roots only, and hence he wrote *peut-il y avoir*.

b) Consider the signs in some polynomial expression, for example

```
+ + - - + - - - + - + -
```

If we multiply this polynomial by the binomial + - (that is the binomial x - a, a positive), we obtain

```
   + + - - + - - - + - + -      (seven changes of sign)
x
   + -
 ___________________________
   + + - - + - - - + - + -
     - - + + - + + + - + - +
 ___________________________
   + ± - ± + - ± ± + - + - +
     ?   ?  .  ? ?
```

Now consider the "worst" possible case, that is the one in which we obtain the least number of changes of sign, and we still must have at least 8, that is at least one more than when we started.

We can state that we have done as follows: we multiplied by x - a (a positive), that is we have constructed a new equation, which has the additional positive root a, and as a consequence we have increased the number of changes of sign by at least one. Hence, a polynomial equation cannot have more positive roots than changes of sign.

A similar argument gives the result for negative roots and non-changes of sign.

7. a) The required change is to alter the sign of every term in an even position in the equation. For example,

$$x^3 + 6x^2 + 11x + 6 = 0$$

has three negative roots, -1, -2, -3 . If we make the required change we have

$$x^3 - 6x^2 + 11x - 6 = 0$$

which has the three positive roots 1, 2 and 3 . The rule is correct on condition that all the powers of x appear in the equation - even if with zero coefficient.

b) Without loss of generality, we may assume that the even positions are occupied by odd powers of x. (For example, the equation in part (a) can be rewritten by adding $0x^4$ at the beginning.) Then changing their sign is equivalent to replacing

$$f(x) = 0$$

by

$$f(-x) = 0$$

and the roots of the latter are of opposite sign to the roots of the former.

8. Although Descartes does not give full recognition to the negative numbers (he refers to false roots and to -5 as the "defect" of the quantity 5), nevertheless there is a distinct advance on the position of Viète. Descartes does refer to negative numbers and deal with them as roots of an equation.

RENE DESCARTES

A. Arcavi
M. Bruckheimer ©

Weizmann Institute
Israel
1983

THE WEIZMANN INSTITUTE OF SCIENCE
REHOVOT · ISRAEL
DEPARTMENT OF SCIENCE TEACHING

The Negative numbers

Worksheet: Contradictions in the use of negative numbers

Part A: Antoine Arnauld

Antoine Arnauld (1612-1694) was a Frenchman, theologian and mathematician. He was close to the mathematician and philosopher B. Pascal (1623-1662). One of Arnauld's best known works is the *Nouveaux Eléments de Géometrie* (1667)*.

The following is a quotation** from a letter by Arnauld addressed to Prestet (1648-1691), in which the central issue is Arnauld's reservations with regard to negative numbers.

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* More details about Arnauld and his work can be found in:
Coolidge J.L., *The Mathematics of Great Amateurs*, Oxford,1949.

** As it is brought in:
Schrecker P., Arnauld, Malebranche, Prestet et la Theorie des Nombres Negatifs, *Thales* 1935,vol.2,82-90.

....je ne comprens pas que le quarré de — 5 puisse être la même chose que le quarré de + 5, et que l'un et l'autre soit + 25. Je ne sçai de plus comment ajuster cela au fondement de la multiplication, qui est que l'unité doit être à l'une des grandeurs que l'on multiplie, comme l'autre est au produit. Ce qui est également vrai dans les entiers et dans les fractions. Car 1 est à 3, comme 4 est à 12. Et 1 est à 1/3, comme 1/4 est à 1/12. Mais je ne puis ajuster cela aux multiplications de deux moins.. Car dira-t-on que + 1 est à — 4, comme — 5 est à + 20 ? Je ne le vois pas. Car + 1 est plus que — 4. Et au contraire — 5 est moins que + 20. Au lieu que dans toutes les autres proportions, si le premier terme est plus grand que le second, le troisiéme doit être plus grand que le quatriéme. .

Questions

1. Explain using modern notation what Arnauld means by the *fondement de la multiplication*.

2. Which properties of negative numbers does Arnauld use in his argument?

3. What is the "contradiction" that he sees in the use of negative numbers?

4. How would you answer him?

7

Part B: John Wallis

IOHANNES WALLIS D·D

GEOMETRIAE PROF· SAVILIANVS OXONIAE

REGALIS SOCIETATIS LONDINI SODALIS.

John Wallis (1616-1703) was an Englishman, Savilian Professor of Geometry in Oxford, and a founder member of the Royal Society.

Scott (1981)* writes about Wallis:

... more than any other the precursor of the mighty Newton. ... it seems beyond all doubt that amongst his contemporaries at least Wallis was esteemed eminent, ... over a range of subjects which was truly encyclopaedic. Mathematics, mechanics, sound, philology, the phenomena of the tides, even music - in all these his writings give evidence of profound knowledge ...

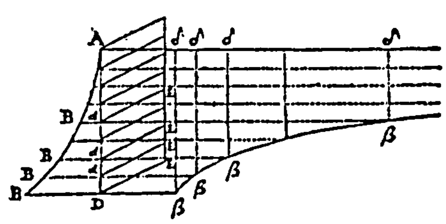
Among Wallis' works are: *Arithmetica Infinitorum* (1655) and his "textbook": *A Treatise on Algebra both Historical and Practical* (1685).

The following is a passage from the *Arithmetica Infinitorum*, page 409:

PROP. CIV.

Theorema.

SI denique ejufmodi Figura AD$\beta\beta$, fic continue decrefcat juxta feriem quæ fit reciproca directæ indicem habenti unitate majorem; habebit illa ad Parallelogrammum infcriptum rationem plufquam infinitam: qualem nempe habere fupponatur numerus pofitivus ad numerum negativum, five minorem nihilo. Nempe eam, quam habet 1 ad indicem unitate auctum.



Puta cum indices feriei Secundanorum, Tertianorum, Quartanorum, &c. fint 2, 3, 4, &c. (unitate majores,) indices ferierum illis reciprocarum erunt -2, -3, -4, &c. qui quamvis unitate augeantur (juxta Prop. 64.) manebunt tamen negativi, puta $-2+1=-1$, $-3+1=-2$, $-4+1=-3$, &c. & propterea ratio quam habet 1 ad indices illos fic auctos, puta 1 ad -1, 1 ad -2, 1 ad -3, &c. major erit quam infinita, five 1 ad 0; quia nempe rationum confequentes funt minores quam 0.

Atque idem continget, fi fumatur reciproca feriei radicum Quadraticarum Tertianorum, Quartanorum, Quintanorum, &c. (cujus indices funt $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c.) vel radicum Cubicarum Quartanorum, Quintanorum, Sextanorum, &c. (cujus indices funt $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c.) aut cuivis denique feriei cujus index eft unitate major. Ut patet.

---

7 * Scott J.F., *The Mathematical Work of John Wallis.* Chelsea Pub. Co., N.Y., 1981.

In this passage Wallis mentions:

... rationem plusquam infinitam: qualem nempe habere supponatur numerus positivus ad numerum negativum, five minorem nihilo.

This means: "a ratio greater than infinity, such as a positive number may be supposed to have to a negative number, or one less than nothing"*.

This conclusion (in modern terminology) is "deduced" from the sequence:

$$\ldots \quad \frac{1}{5} \quad \frac{1}{4} \quad \frac{1}{3} \quad \frac{1}{2} \quad \frac{1}{1} \quad \frac{1}{0} \quad \frac{1}{-1} \quad \frac{1}{-2} \quad \frac{1}{-3} \quad \ldots$$

in which the ratios increase from the first to the sixth, i.e. from $\frac{1}{5}$ to $\frac{1}{0}$, which "was shown" to be infinite*.

So if one continues to the right, one gets:

$$\frac{1}{0} < \frac{1}{-1} < \frac{1}{-2} \quad \ldots$$

and this means that the ratio of a positive to a negative is greater than infinity.

Questions

5. Write the general term of the sequence and draw its graph (as function of n).

6. How would you answer Wallis' argument?

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* See Scott ibid. p. 44-45.

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The Negative numbers

Answer sheet: Contradictions in the use of negative numbers

Part A: Antoine Arnauld

1. Arnauld says that a basic principle of multiplication (of two factors) is that the ratio of unity to one factor is equal to the ratio of the second factor to the product; i.e., in the product ab

$$\frac{1}{a} = \frac{b}{ab} \quad \text{or} \quad \frac{1}{b} = \frac{a}{ab}$$

2. In his argument, Arnauld uses:

   i) the assumption of order as for example in

   $$1 > -4 \ , \quad -5 < 20$$

   ii) (implicitly) the law of signs in multiplication, and specifically a negative times a negative (or alternatively the law of signs in division, if we regard the ratios 1 to -4 and -5 to 20 as such).

3. Arnauld states that if one accepts that the multiplication of two minuses is a plus, for instance

$$(-4)\cdot(-5) = +20 \ ,$$

then the "basic principle of multiplication" (see Qu. 1) implies that

$$\frac{1}{-4} = \frac{-5}{20}$$

that is to say that, a bigger number is to a smaller (1 to -4) as a smaller number is to a bigger (-5 to 20) and this contradicts the proportion concept:

Si le premier terme est plus grand que le second, le troisième doit être plus grand que le quatrième.

4. Before commenting Arnauld's "contradiction" in modern terms, we bring two answers as given by Arnauld's contemporaries.

   i) J. Prestet * answered to Arnauld's letter, in which he says:

Il est clair que le plus et le moins des grandeurs égales se font l'un à l'autre des retranchements réciproques, aprés lesquels il ne reste rien, c'est-à-dire que comme le plus d'une grandeur en détruit et retranche le moins, le moins pareillement en détruit et retranche le plus. Que + 1 par exemple retranche — 1, comme — 1 retranche + 1 ; ou ce qui signifie la même chose, que + 1 — 1 = 0, comme — 1 + 1 = 0, Il est donc égal de dire le plus d'une grandeur, ou le retranchement de son moins. Or le retranchement du moins signifie encore la même chose que moins moins. Il sera donc égal de dire plus, ou de dire moins moins. Et ainsi ce sera la même chose de dire + 5 fois 5, ou + 25 ; ou de dire — — 5 fois 5, ou — — 25. Or — — 25 est le quarré de — 5. Car multiplier — 5 par — 5, c'est prendre — 5 autant de fois qu'il y a d'unitez dans — 5, c'est à dire — 5 fois : ou ce qui est égal, c'est ôter — 5 autant de fois qu'il y a d'unitez dans 5 ; je ne dis pas — 5, parce que ce moins est transplacé dans le mot d'*ôter* qui le signifie. Or je ne puis ôter une fois — 5, qu'en mettant une fois + 5. Et ainsi je ne puis ôter 4 fois — 5, qu'en mettant 4 fois + 5 ou + 20. Cela paroît démonstratif : et c'est de quoi je voulois établir le principe dans ma lettre, lorsque j'y marquois que si un homme qui n'a rien et qui ne doit rien, recevait la diminution ou le retranchement d'une dette supposée de 1000 écus, ou que

* According to Schreker, ibid.

8

si on luy ôtoit — 1000 écus, il auroit ensuite + 1000 écus, parce qu'on n'auroit pû luy ôter cette dette qu'en luy donnant + 1000 écus de quoy la payer. Et comme on suppose qu'il ne doit rien, il seroit en droit de garder ce qu'il auroit receu. De sorte qu'il auroit véritablement 1000 écus.

« Si donc + 1 détruit — 1, comme — 1 détruit +1 ; ou si + 1 est une fois ôté ou retranché dans — 1, comme — 1 une fois ôté ou retranché dans + 1 : ces termes *une fois ôté ou retranché*, et l'expression plus abrégée — 1, exposeront également ce que + 1 est à — 1, et ce que — 1 est à + 1. Il est assez évident par les *Nouveaux Elémens de Géométrie* que plus en moins, ou moins en plus, donne également moins. Ils semblent même supposer, comme une vérité constante, qu'on aura prouvé que moins en plus donne moins, lorsqu'on aura prouvé que plus en moins donne moins. Il sera donc constant par vos propres principes, que + 1 divisé par — 1, et que — 1 divisé par + 1, donneront un même exposant — 1 ; ou que + 1 est au juste contenu dans — 1, comme — 1 dans + 1. Or il est clair que partout où les exposants sont les mêmes, les rapports sont égaux ; les différences ne sont pas l'essentiel des proportions géométriques, elles le sont seulement des arithmétiques. Il suffit pour la géométrique que le premier terme soit dans le second de la même sorte que le troisiéme est dans le quatriéme. Cette notion est si étendue qu'elle embrasse les entiers, les fractions, les incommensurables, et les grandeurs négatives et positives. Il est donc permis de raisonner ainsi, + 1 : — 1 = — 1 : + 1. Or + 1 : — 1 = + 5 : — 5 = + 4 : — 4. Et — 1 : + 1 = — 5 : + 5 = — 4 : + 4. Donc + 1 : — 1 = — 5 : + 5 = — 4 : + 4. Et par un changement alterne, + 1 : — 5 = — 1 : + 5. Et + 1 : — 4 = — 1 : + 4. Or — 1 : + 5 = — 5 : + 25. Et — 1 : + 4 = — 5 : + 20. Donc par égalité + 1 : — 5 = — 5 : + 25. Et + 1 : — 4 = — 5 : + 20. Et s'il est vrai que l'unité est ôtée ou retranchée 5 fois dans — 5, et 4 fois dans — 4, comme — 5 est ôté ou retranché 5 fois dans + 25, et 4 fois dans + 20 ; on trouvera encore en divisant + 25 par — 5, et — 5 par + 1, un même exposant — 5 ; et en divisant + 20 par — 5, et — 4 par + 1, un même exposant — 4. De sorte qu'il est clair en toutes manières, que moins multiplié par moins donne plus, essentiellement et par luy-même.

ii) G.W. Leibniz (1646-1716), the German mathematician who was the "co-founder" of the calculus wrote an answer to Arnauld's argument in 1712*, in which he agreed that these are not truly ratios [*Veras illas ratione non esse*]. But he argued that one can calculate with these ratios as well as one does with imaginary quantities, so they are tolerable [*toleranter verae*] and useful,

* Leibniz G.W., *Math. Schriften*, Vol. 5, Ch. 29, p.387-389.

[*habent tamen usum magnum in calculando*] in spite of their lack of rigor [*Rigorem quidem non sustinent*].

The following is the full quotation:

XXIX.

OBSERVATIO QUOD RATIONES SIVE PROPORTIONES NON HABEANT LOCUM CIRCA QUANTITATES NIHILO MINORES, ET DE VERO SENSU METHODI INFINITESIMALIS. *)

Cum olim Parisiis Vir summus *Antonius Arnaldus* sua nova Geometriae Elementa mecum communicaret, atque in iisdem admirari se testatus fuisset, quomodo posset esse 1 ad -1, ut -1 ad 1, quae res probari videtur ex eo, quod productum est idem sub extremis quod sub mediis, cum utroque prodeat $+1$; jam tum dixi mihi videri, *veras* illas *rationes* non esse, in quibus quantitas nihilo minor est antecedens, vel consequens, etsi in calculo haec, ut alia *imaginaria*, tuto et utiliter adhibeatur. Et sane identitatis rationum verarum fundamentum est rerum similitudo, quae facit exempli causa, ut segmentis similibus diversorum circulorum assumtis sit ubique eadem ratio chordae ad radium, seu ut chorda minoris se habeat ad radium minoris, vel ut chorda majoris ad radium majoris. Sed vero nulla plane apparet similitudo in supradicta Analogia. Si enim -1 est minus nihilo, utique 1 ad -1 erit ratio majoris ad minus; sed vero contra ratio -1 ad 1 est ratio minoris ad majus; quomodo ergo utrobique eadem ratio erit? Sed rationes istas esse imaginarias, etiam alio certissimo argumento comprobabo, scilicet a Logarithmis. Nempe ratio, cui nullus datur respondens Logarithmus, ratio vera non est. Porro posito unitatis Logarithmum esse 0, rationis -1 ad 1 idem est Logarithmus, qui ipsius -1; at ipsius -1 non datur Logarithmus. Non enim est positivus, nam talis omnis est Logarithmus numeri positivi unitate majoris. Sed tamen etiam non est negativus, quia talis omnis est Logarithmus numeri positivi unitate minoris. Ergo Logarithmus ipsius -1 cum nec positivus sit nec negativus, superest ut sit non verus, sed imaginarius. Itaque et ratio, cui respondet, *non vera*, sed *imaginaria* erit. Idem etiam sic probo: Si daretur verus Logarithmus ipsius -1, seu rationis -1 ad 1, ejus logarithmi dimidium foret Logarithmus ipsius $\sqrt{-1}$, sed $\sqrt{-1}$ est quantitas imaginaria. Itaque daretur Logarithmus verus imaginariae quantitatis, quod est absurdum. Et proinde nonnihil humani passus est insignis in paucis Geometra *Johannes Wallisius*, cum dixisset rationem 1 ad -1 esse plus quam infinitam; et recte hoc (etsi aliis considerationibus) celeberimus *Varignonius*
8 rejicit. Interim nolim cum ipso negare, -1 esse quantitatem

nihilo minorem, modo id sano sensu intelligatur. Tales enuntiationes sunt *toleranter verae*, ut ego cum summo Viro *Joachimo Jungio* loqui soleo; Galli appellarent *passables*. Rigorem quidem non sustinent, habent tamen usum magnum in calculando, et ad artem inveniendi, universalesque conceptus valent. Talis fuit locutio Euclidis, cum Angulum contactus dixit esse rectilineo quovis minorem; tales sunt multae Geometrarum aliae, in quibus est figuratum quodammodo et crypticum dicendi genus. Sunt tamen quidem, ut sic dicam, *tolerabilitatis*. Porro, ut nego rationem, cujus terminus sit quantitas nihilo minor, esse realem, ita etiam nego, proprie dari numerum infinitum vel infinite parvum, etsi Euclides saepe, sed sano sensu, de linea infinita loquatur. *Infinitum continuum* vel *discretum* proprie nec unum, nec totum, nec quantum est, et si analogia quaedam pro tali a nobis adhibeatur, ut verbo dicam, est modus loquendi; cum scilicet plura adsunt, quam ullo numero comprehendi possunt, numerum tamen illis rebus attribuemus analogice, quem infinitum appellamus. Itaque jam olim judicavi, cum infinite parvum esse errorem dicimus, intelligi dato quovis minorem, revera nullum; et cum ordinarium, et infinitum, et infinities infinitum conferimus, perinde esse ac si conferremus ascendendo diametrum pulvisculi, diametrum terrae, et diametrum orbis fixarum, aut his quantumvis (per gradus) majora minoraque, eodemque sensu descendendo diametrum orbis fixarum, diametrum terrae, et diametrum pulvisculi posse comparari ordinario, infinite parvo, et infinities infinite parvo, sed ita ut quodvis horum in suo genere quantumvis majus aut minus concipi posse intelligatur. Cum vero saltu ad ultimum facto ipsum infinitum aut infinite parvum dicimus, commoditati expressionis seu breviloquio mentali inservimus, sed non nisi *toleranter vera* loquimur, quae explicatione *rigidantur*. Atque haec etiam mea sententia est de areis illis Hyperboliformium Asymptoticis, quae infinitae, infinitiesque infinitae esse dicuntur, id est talia rigorose loquendo vera non esse posse, tamen sano aliquo sensu tolerari. Atque haec tum ad terminandas virorum clarissimorum Varignonii et Grandii controversias, tum ad praecavendos chimericos quosdam conceptus, tum denique ad elidendas oppositiones contra methodum *infinite similem* prodesse possunt.

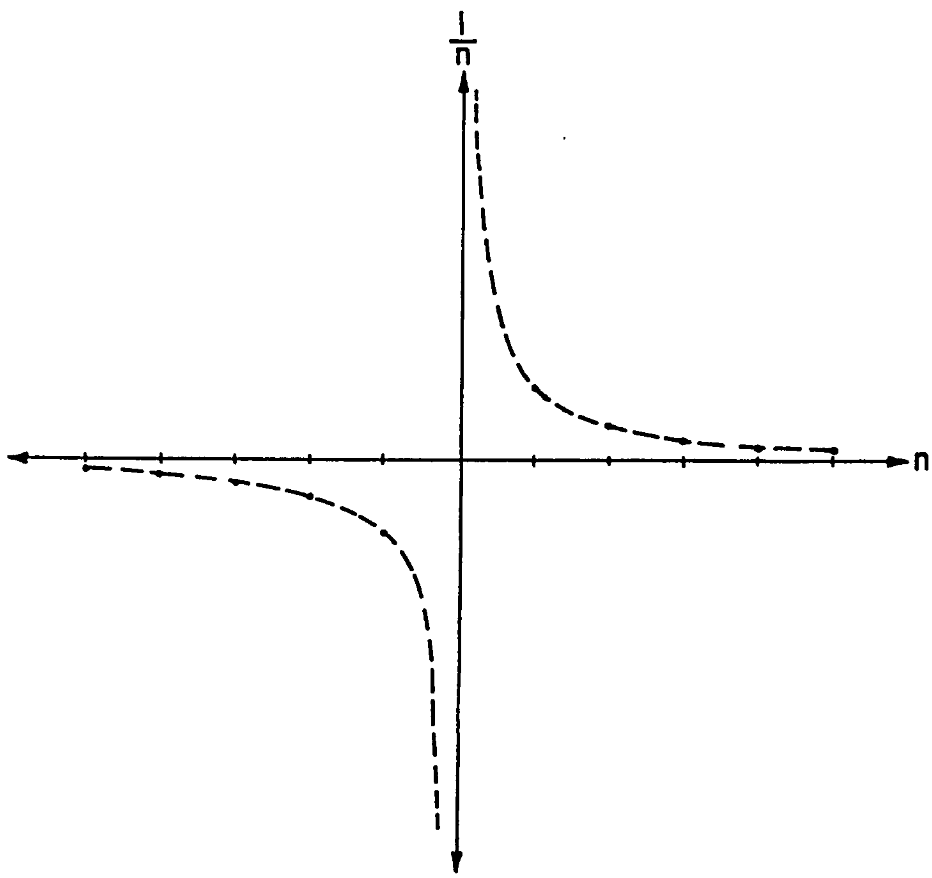
In modern terms we can answer Arnauld as follows. The ratio concept and the concept of "larger" and "smaller" (i.e. order) to Arnauld are those to which he was used in the set of positive numbers. Now he extends these concepts to a larger set (i.e. includes the negative numbers), and there is no a priori reason why these properties can be carried over without creating contradictions.

In our case the relationship between order and proportion is a property of positive numbers only: we cannot have them both in the enlarged set. Hence if we chose to maintain the order concept, in the form to which we are used, the idea of ratio of smaller to larger must go. Another example is the following: in the set of positive numbers $a < 1 \Rightarrow ab < b$. This property does not carry over to the set of all numbers. In a subsequent worksheet (Peacock) we shall deal with this point at length.

8

Part B: John Wallis

5. The general term of the sequence in Wallis' argument is $\frac{1}{n}$, and its graph is:



6. The great Swiss mathematician Leonard Euler (1707-1783) was one of the most prolific of all time*. In his book *Differenzial Rechnung* Euler answered Wallis argument. The following is a quotation from the 1790 German edition.

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* More details about Euler can be found in the Euler worksheet in this series.

§. 100.

Auf eben die Art trifft man in den Reihen öfters unendlich große Glieder an, z. B. in der harmonischen Reihe, deren allgemeines Glied $\frac{1}{x}$ ist, wo zu dem Anzeiger x = o das unendlich große Glied $\frac{1}{o}$ gehört, und die ganze Reihe folgende ist:

$$\text{rc.} - \frac{1}{4} - \frac{1}{3} - \frac{1}{2} - \frac{1}{1} + \frac{1}{o} + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{rc.}$$

Geht man daher von der Rechten gegen die Linke zu, so wachsen die Glieder, so daß $\frac{1}{o}$ unendlich groß ist, aber so bald die Glieder über diese Grenze kommen, so werden sie negativ und nehmen ab. Hiernach kann man also die unendlich große Größe als eine Grenze betrachten, jenseits welcher die positiven Zahlen negativ werden, und umgekehrt. Dies hat einige verleitet, zu behaupten, daß die negativen Zahlen als Zahlen, die größer seyn als das Unendliche, betrachtet werden könnten, weil die beständig wachsenden Glieder dieser Reihe, nachdem sie das Unendliche erreicht haben, negativ werden. Aber wenn man die Reihe nimmt, deren allgemeines Glied $\frac{1}{xx}$ ist, so werden die Glieder derselben nach dem Uebergange durchs Unendliche wieder positiv

$$\text{rc.} \frac{1}{9} + \frac{1}{4} + \frac{1}{1} + \frac{1}{o} + \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \text{rc.}$$

und davon wird doch wohl Niemand behaupten wollen, daß sie größer sind als das Unendliche.

In summary, he states that from sequences of reciprocals one cannot conclude anything. He exemplifies his rejection of the Wallis argument with $\frac{1}{n^2}$,

$$\ldots, \frac{1}{9}, \frac{1}{4}, \frac{1}{1}, \frac{1}{0}, \frac{1}{1}, \frac{1}{4}, \frac{1}{9}, \ldots$$

from which "no one will assert that they (the positive terms on the right) are greater than infinity".

Euler rubs in the point that this form of argument only leads to contradictions, by giving another example

§. 101.

Oft macht auch das Unendliche in den Reihen die Grenze zwischen den reellen und imaginären Gliedern, wie in der, deren allgemeines Glied $\frac{1}{\sqrt{x}}$ ist:

$$\text{etc.} + \frac{1}{\sqrt{-3}} + \frac{1}{\sqrt{-2}} + \frac{1}{\sqrt{-1}} + \frac{1}{0} + \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \text{etc.}$$

indeß folgt hieraus nicht, daß die imaginären Größen größer seyen als das Unendliche, denn aus der vorhin angeführten Reihe

$$\text{etc.} + \sqrt{-3} + \sqrt{-2} + \sqrt{-1} + 0 + \sqrt{1} + \sqrt{2} + \sqrt{3} + \text{etc.}$$

würde eben so gut folgen, daß sie kleiner wären.

From the first sequence one would include that the imaginary numbers are greater than infinity, and from the second that they are smaller.

In modern terms, we would say that Wallis' argument is invalid, because it assumes implicitly that the sequence $n \longrightarrow \frac{1}{n}$ is continuous, which it clearly is not, as the graph in answer to Qu. 5 shows. There is a discontinuity at 0. The function is decreasing for all n, except at n = 0.

Leonhard Euler's

Vollständige Anleitung

zur

Differenzial-Rechnung.

Aus dem Lateinischen übersetzt

und

mit Anmerkungen und Zusätzen begleitet

von

Johann Andreas Christian Michelsen,

Professor der Mathematik und Physik am Berlinischen Gymnasium.

Erster Theil.

Berlin und Libau,

bey Lagarde und Friedrich 1790.

Front page of Euler's *Differenzial Rechnung*.

THE WEIZMANN INSTITUTE OF SCIENCE
REHOVOT · ISRAEL
DEPARTMENT OF SCIENCE TEACHING

The Negative numbers

Worksheet: Nicholas Saunderson

THE

ELEMENTS of ALGEBRA,

IN TEN BOOKS:

By NICHOLAS SAUNDERSON *LL.D.*
Late *Lucasian* Professor of the Mathematics in the University of CAMBRIDGE, and Fellow of the Royal Society.

VOLUME THE FIRST,
Containing the Five first Books.

To which are prefixed

I. The LIFE and CHARACTER of the AUTHOR.
II. His PALPABLE ARITHMETIC Decyphered.

CAMBRIDGE,
Printed at the UNIVERSITY-PRESS:
MDCCXLI.

The English mathematician, Nicholas Saunderson, was born in 1682 and died in 1739. At the age of one year he contracted smallpox and became blind. This tragedy did not, however, prevent him from studying, teaching and living a creative life. He learnt Greek, Latin and French, and "heard" the works of Euclid, Archimedes and Diophantus in the original, learning some parts by heart.

His father taught him arithmetic and he was very skilled in solving problems and executing long calculations, using his prodigious memory and his creation "palpable arithmetic", a nail board for doing arithmetic and creating geometric shapes with silk threads - the forerunner of the modern geoboard. Saunderson made a reputation as an excellent teacher at Cambridge, where he occupied the Lucasian chair, some years previously occupied by Newton.*

The following is an extract from his *The Elements of Algebra*, which was published posthumously in 1741.

---

* Further details on the man and his work can be found in the introduction to his book:
Saunderson, N., *The Elements of Algebra*, Vol. 1, Cambridge, 1741.

Of the multiplication of algebraic quantities.

And first, how to find the sign of the product in multiplication, from those of the multiplicator and multiplicand given.

5. Before we can proceed to the multiplication of algebraic quantities, we are to take notice, that if the signs of the multiplicator and multiplicand be both alike, that is, both affirmative, or both negative, the product will be affirmative, otherwise it will be negative: thus $+4$ multiplied into $+3$, or -4 into -3 produces in either case $+12$: but -4 multiplied into $+3$, or $+4$ into -3 produces in either case -12.

If the reader expects a demonstration of this rule, he must first be advertised of two things: *first*, that numbers are said to be in arithmetical progression, when they increase or decrease with equal differences, as 0, 2, 4, 6; or 6, 4, 2, 0; also as $3, 0, -3$; $4, 0, -4$; $12, 0, -12$; or $-12, 0, +12$: whence it follows, that three terms are the fewest that can form an arithmetical progression; and that of these, if the two first terms be known, the third will easily be had: thus if the two first terms be 4 and 2, the next will be 0; if the two first be 12 and 0, the next will be -12; if the two first be -12 and 0, the next will be $+12$, &c.

2dly, If a set of numbers in arithmetical progression, as 3, 2 and 1, be successively multiplied into one common multiplicator, as 4, or if a single number, as 4, be successively multiplied into a set of numbers in arithmetical progression, as 3, 2 and 1, the products 12, 8 and 4, in either case, will be in arithmetical progression.

This being allowed, (which is in a manner self-evident,) the rule to be demonstrated resolves itself into four cases:

1st, That $+4$ multiplied into $+3$ produces $+12$.

2dly, That -4 multiplied into $+3$ produces -12.

3dly, That $+4$ multiplied into -3 produces -12.

And *lastly*, that -4 multiplied into -3 produces $+12$. These cases are generally expressed in short thus: first $+$ into $+$ gives $+$; secondly $-$ into $+$ gives $-$; thirdly $+$ into $-$ gives $-$; fourthly $-$ into $-$ gives $+$.

Case 1st. That $+4$ multiplied into $+3$ produces $+12$, is self-evident, and needs no demonstration; or if it wanted one, it might receive it from the first paragraph of the 3d article; for to multiply $+4$ by $+3$ is the same thing as to add $4+4+4$ into one sum; but $4+4+4$ added into one sum give $+12$, therefore $+4$ multiplied into $+3$, gives $+12$.

Case 2d. And from the second paragraph of the 3d art. it might in like manner be demonstrated, that -4 multiplied into $+3$ produces -12: but I shall here demonstrate it another way, thus: multiply the terms of this arithmetical progression $4, 0, -4$, into $+3$, and the products will be in arithmetical progression, as above; but the two first products are 12 and 0; therefore the third will be -12; therefore -4 multiplied into $+3$, produces -12.

Case 3d. To prove that $+4$ multiplied into -3 produces -12; multiply $+4$ into $+3$, 0, and -3 successively, and the products will be in arithmetical progression; but the two first are 12 and 0, therefore the third will be — 12; therefore + 4 multiplied into — 3 produces — 12.

Case 4th. Laſtly, to demonſtrate, that — 4 multiplied into — 3 produces + 12, multiply — 4 into 3, 0, and — 3 ſucceſſively, and the products will be in arithmetical progreſſion; but the two firſt products are — 12 and 0, by the ſecond caſe; therefore the third product will be + 12; therefore — 4 multiplied into — 3 produces + 12.

Caſ. 2*d.*	+4,	0,	−4
	+3,	+3,	+3
	+12,	0,	−12.

Caſ. 3*d*,	+4,	+4,	+4
	+3,	0,	−3
	+12,	0,	−12.

Caſ. 4*th*,	−4,	−4,	−4
	+3,	0,	−3
	−12,	0,	+12.

Theſe 4 caſes may be alſo more briefly demonſtrated thus: + 4 multiplied into + 3 produces + 12; therefore — 4 into + 3, or + 4 into — 3 ought to produce ſomething contrary to + 12, that is, — 12: but if — 4 multiplied into + 3 produces — 12, then — 4 multiplied into — 3 ought to produce ſomething contrary to — 12, that is, + 12; ſo that this laſt caſe, ſo very formidable to young beginners, appears at laſt to amount to no more than a common principle in Grammar, to wit, that two negatives make an affirmative; which is undoubtedly true in Grammar, though perhaps it may not always be obſerved in languages.

Questions

1. Before giving a demonstration of the various cases of the multiplication rule, Saunderson notes two things. What is the difference between them?

2. Reread the extract and note down all the definitions, assumptions and all the conclusions.

3. What is the difference between the first demonstration of the four cases and the second?

4. In what way are Saunderson's demonstrations different from and in what way similar to, those given in the school text you use? (In the Hebrew version, this question is specifically directed at the teaching texts used in the Rehovot programme.)

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1983

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The Negative numbers

Answer sheet: Nicholas Saunderson

1. *Firstly,* Saunderson *defines* an arithmetic progression, and gives examples.

   Secondly, he makes a basic *assumption* (axiom), on which he bases his proofs of the various cases. The axiom is that if we multiply an arithmetic progression by a number, or a number by an arithmetic progression, the result is another arithmetic progression.
   In the case where the terms of the A.P. are positive and the multiplier is also positive, we do not really need the assumption, and Saunderson brings an example. But his assumption extends to the case where there are also negative terms.

   Mathematically one may object that his assumption is "as big as" that which he is trying to prove, but *didactically* there is something to be said for it - see answer to question 4.

2.

Definitions	Assumptions	Conclusions
Numbers are said to be in arithmetic progression, when they increase or decrease with equal differences	*If a set of numbers in arithmetical progression be successively multiplied into one common multiplicator, or if a single number be successively multiplied into a set of numbers in arithmetical progression, the products, in either case, will be in arithmetical progression.**	+3•+4 = +12 +3•-4 = -12 -3•+4 = -12 -3•-4 = +12
	If we replace one term of a product by its negative (*contrary*), the product also becomes the negative (*contrary*) of the original.	As above

* For the one case of "positive x positive" this is not an assumption, but can be "proved" from earlier work with products of positive numbers. But for all the other cases it is used as an assumption on which to base the "demonstration".

3. As Saunderson himself remarks, the second demonstration is given in brief. We assume that his objective was didactical: to reinforce the result obtained in the first demonstration by looking at it from another point of view.
In the first demonstration, he very carefully states his assumption, and makes explicit its use in each case.
In the second, the assumption (as given in the answer to Qu. 2) is given implicitly as part of the demonstration.

To this he adds the suggestive remark from grammar. (See also answer to Qu. 4.)

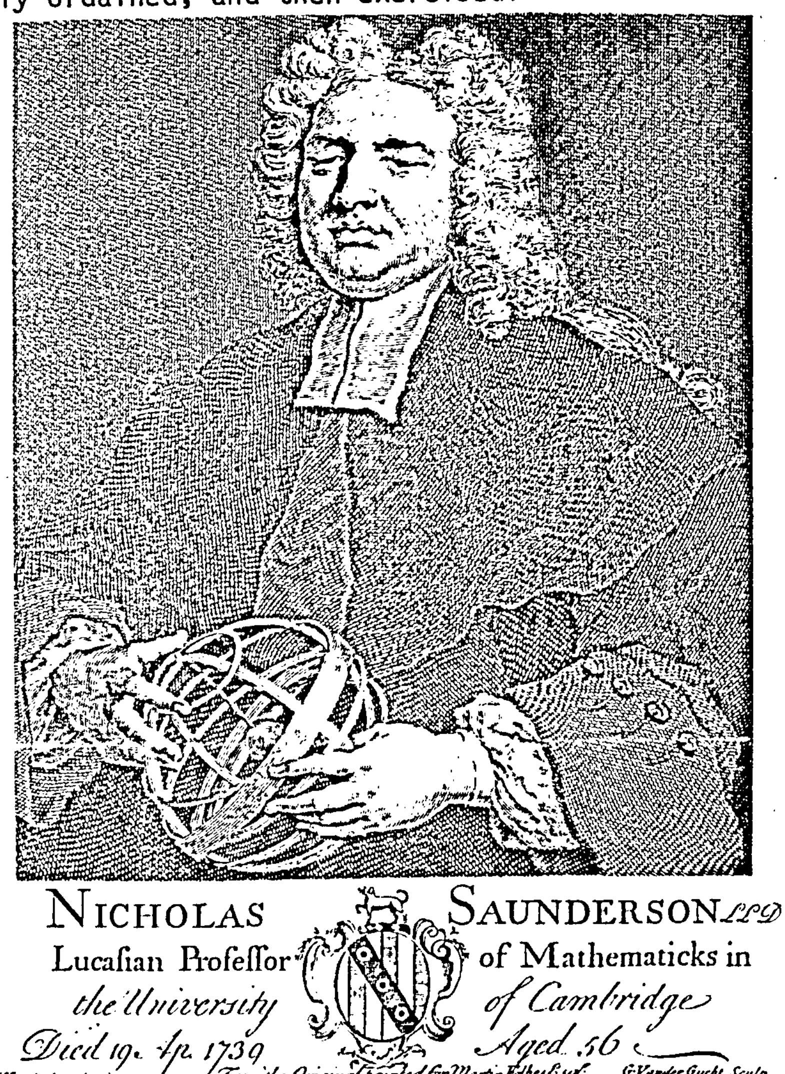
4. Clearly the answer depends on the text in use. In the Rehovot programme there are three parallel texts designed to be used with different ability levels. For the top stream the approach is "mathematical" in that the basic assumption is that the basic laws, such as distributively of multiplication over addition, carry over from the set of positive numbers to the set of all numbers. The properties dealt with in the extract are then proved*. There is a qualitative mathematical difference between this approach and that of Saunderson, in that we assume a general mathematical property to hold, rather than a very particular one.

   For the two other streams the approach has some affinity with the first demonstration by Saunderson, in that descending arithmetic sequences of positive numbers are multiplied by a positive number. The sequence is then extended to negative numbers and the students are asked to complete the sequence of products intuitively, and so "discover" the properties of multiplication. By suitable choices of sequences, all the results can be gradually obtained.

The difference between this approach and that of Saunderson is that in the Rehovot texts, the approach is entirely intuitive, whereas Saunderson "demonstrates".

*In the Worksheet George Peacock (qu. 4) we bring this approach as it appears in SMSG.

Saunderson's second demonstration has no parallel in the Rehovot texts. Didactically it may be regarded as adding a little intuition to the approach to be found in some older texts, in which the bare rules are stated, as if divinely ordained, and then exercised.



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1983

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The Negative Numbers

Worksheet: Leonard Euler

It is maintained that Leonard Euler (1707 - 1783) was the most prolific of all mathematicians. Even total blindness in the last seventeen years of his life did not diminish his creativity. He made contributions to analysis (which was then the relatively new "creation" of Newton and Leibniz), algebra, trigonometry, number theory and many other fields.

Euler was born in Basel, and against the wishes of his father, who was a theologian, he turned to mathematics. At the age of 17 he already received a degree in mathematics, and at 19 published his first paper. Following the brothers Bernoulli (part of a family of Swiss mathematicians), he moved to St. Petersburg, where he married and had 13 children. In 1740 he moved to the academy in Berlin, returning to St. Petersburg in 1766.*

Among his many creations are several "textbooks" including the *Elements of Algebra*. This book was written (or better, dictated) by Euler after he went blind and was published in 1770. It was translated into several languages and continued to be published for some 150 years - we have seen an edition published in German during the first world war, and maybe this wasn't the last.
The following extract is taken from an English (fourth) edition published in 1828.

*For further details on the man and his contribution to mathematics see Boyer, C.B., *A History of Mathematics*. J. Wiley & Sons, 1969.

11

31. Hitherto we have considered only positive numbers; and there can be no doubt, but that the products which we have seen arise are positive also: viz. $+a$ by $+b$ must necessarily give $+ab$. But we must separately examine what the multiplication of $+a$ by $-b$, and of $-a$ by $-b$, will produce.

32. Let us begin by multiplying $-a$ by 3 or $+3$. Now, since $-a$ may be considered as a debt, it is evident that if we take that debt three times, it must thus become three times greater, and consequently the required product is $-3a$. So if we multiply $-a$ by $+b$, we shall obtain $-ba$, or, which is the same thing, $-ab$. Hence we conclude, that if a positive quantity be multiplied by a negative quantity, the product will be negative; and it may be laid down as a rule, that + by + makes + or *plus*; and that, on the contrary, + by −, or − by +, gives −, or *minus*.

33. It remains to resolve the case in which − is multiplied by −; or, for example, $-a$ by $-b$. It is evident, at first sight, with regard to the letters, that the product will be ab; but it is doubtful whether the sign +, or the sign −, is to be placed before it; all we know is, that it must be one or the other of these signs. Now, I say that it cannot be the sign −: for $-a$ by $+b$ gives $-ab$, and $-a$ by $-b$ cannot produce the same result as $-a$ by $+b$; but must produce a contrary result, that is to say, $+ab$; consequently, we have the following rule: − multiplied by − produces +, that is, the same as + multiplied by + *.

Questions

1. The multiplication of (real) numbers is divided into four cases

   $+\cdot+$, $+\cdot-$, $-\cdot+$ and $-\cdot-$.

   How does Euler deal with each of these?

2. Find the hidden assumptions in the given extract (there are at least three).

3. The extract is taken from a "textbook". Suppose that you were to use this text in your class, and discuss its strengths and weaknesses.

4. What is the difference between Euler's arguments and Saunderson's ?

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The Negative Numbers

Answer sheet: Leonard Euler

1. i) The case +•+ is self evident - "+a by +b must necessarily give +ab".

   ii) The two cases +•- and -•+ are dealt with together. The explanation uses a model - that of a debt to represent the negative quantity. The model does not distinguish between +•- and -•+ .

   iii) In the fourth case, -•- , "it is evident at first sight" that the product of -a and -b must be of magnitude ab. It remains only to determine whether we should prefix + or -. Euler resolves this by "contradiction": (-a)•(+b) and (-a)•(-b) cannot have the same sign (why Euler does not say), hence the assumption that the latter is equal to -ab leads to a contradiction. Hence (-a)•(-b) = +ab.

Thus Euler uses three essentially different types of "argument" to resolve the four cases:

(i) "it is clear that" ;
(ii) model to suggest the general result ;
(iii) mathematical-intuitive argument.

2. Euler's assumptions are all implicit:

i) Transfer from the algebraic problem to a model (debt) and back again, as a legitimate process in reaching a mathematical conclusion.

ii) The extension of the commutative law, from multiplication of positive numbers to that of negative numbers. Thus Euler exemplifies only "negative times positive".

iii) The assumption that the numerical value of $(-a)\cdot(-b)$ must be ab.

iv) The assumption that $(-a)\cdot(+b)$ and $(-a)\cdot(-b)$ cannot have the same sign. This assumption itself may have been tacitly based on a more fundamental assumption, i.e. that the *cancellation law* can be extended from the positive to the negative numbers. Thus if we assume that

$$(-a)\cdot(+b) = (-a)\cdot(-b)$$

and that the cancellation law holds, this implies that

$$+b = -b$$

which is false if $b \neq 0$.

3. Various points arise in this extract:

i) The section is didactically well organised. There is an introduction, from an example, the general case is "deduced". At the end of each sub-section there is an interim summary, and each new case is introduced.

12

ii) We have already remarked in the answer to Qu. 2 on some assumptions - the didactical validity of these assumptions is dependent on the mathematical maturity of the students.
Another point arising is the fact that Euler (in his extract) regards the variable a as representing a positive quantity. (Histories of mathematics credit the Dutch mathematician, Johann Hudde (1629-1704) with being the first to use a letter to represent a positive or negative quantity.) There is a "didactical danger" to the present day in this, since students may tend to assume in other contexts that a always represents a positive quantity and that $-a$ is always negative.

iii) Euler writes: "... since -a may be considered as a debt, it is evident that if we take the debt three times, it must become three times greater, and consequently the required product is -3a".
This may well give the impression that -3a is greater than -a, which is not true (if a is positive). The order relation in the negative numbers, e.g. that -3 is bigger than -4, often causes difficulty, and one should avoid possibilities of reinforcing false impressions.

iv) In short tests on negative numbers given to some grade 7 classes we found an interesting (if not altogether serious) mistake. A few students justified

$$(-4)\cdot(+3) = (+4)\cdot(-3)$$

by recourse to the commutative law.

This may well arise from our (and Euler's) imprecise (if suggestive) phrasing in such results as

"... + by - , or - by + , gives - " .

It would be presumptive if we were to leave the impression that we are criticising Euler. In this question, we have used the text as an object on which to having a discussion of some didactical points arising in the introduction of the multiplication of negative numbers. Most treatments have their pros and cons and some compromise is usually necessary. It is, however, of more than academic interest to be aware of any mistakes we may be inducing by our teaching, in order that we may plan counter action.

As far as Euler is concerned, we should remember that his text was very popular and went through many editions, in English, French, Russian and German. His text belongs to an era of algebra textbooks, as described in the following*:

*Boyer C.B., *A History of Mathematics*, 1968, Wiley & sons.

The leading Continental mathematicians of the mid-eighteenth century were primarily analysts, but we have seen that their contributions were not limited to analysis. D'Alembert had given an imperfect proof of the fundamental theorem of algebra, and Clairaut in 1740 had published a textbook, *Élémens d'algèbre*, which was so popular it went through a sixth edition in 1801. Euler not only contributed to the theory of numbers, but also composed a popular algebra textbook that appeared in German and Russian editions at St. Petersburg in 1770–1772, in French (under the auspices of d'Alembert) in 1774, and in numerous other versions, including American editions in English. The exceptionally didactic quality of Euler's *Algebra* is attributed to the fact that it was dictated by the blind author through a relatively untutored domestic. The textbooks of Clairaut and Euler were not widely used in England, in part because of British mathematical isolationism during the later eighteenth century and in part because Maclaurin and others had composed good textbooks on an elementary level. Maclaurin's *Treatise of Algebra* went through half a dozen editions from 1748 to 1796. A rival *Treatise of Algebra* by Thomas Simpson (1710–1761) boasted at least eight editions at London from 1745 to 1809; another, *Elements of Algebra*, by Nicholas Saunderson (1682–1739), enjoyed five editions between 1740 and 1792. Simpson was a self-taught genius who won election to the Royal Society in 1745 but whose turbulent life ended in failure half a dozen years later. His name nevertheless is preserved in the so-called Simpson's rule, published in his *Mathematical Dissertations on Physical and Analytical Subjects* (1743), for approximate quadratures using parabolic arcs; but this result had appeared in somewhat different form in 1668 in the *Exercitationes geometricae* of James Gregory. Saunderson's life, by contrast, was an example of personal triumph over an enormous handicap—total blindness from the age of one, resulting from an attack of smallpox.

Algebra textbooks of the eighteenth century illustrate a tendency toward increasingly algorithmic emphasis, while at the same time there remained considerable uncertainty about the logical bases for the subject. Most authors felt it necessary to dwell at length on the rules governing multiplications of negative numbers, and some rejected categorically the possibility of multiplication of two negative numbers. The century was, par excellence, a textbook age in mathematics, and never before had so many books appeared in so many editions.

4. In his first approach, Saunderson deals with each of the four cases separately, deducing each one (except, possibly, the first) from his basic assumption that a number multiplied by an arithmetic progression, or an arithmetic progression by a number, gives an arithmetic progression. Euler, on the other hand, makes a number of assumptions (as detailed in the answer to Qu. 2), using the "most convenient" in each case.

In his second approach, Saunderson again uses a single argument, which is similar to that of Euler in the case $- \cdot -$. In this connection, it is interesting to quote Smith*, who remarks that in the "old textbooks" there is a great effort to "prove" the "law of signs", and then continues:

> ...The favorite one of these "proofs" was this: Since multiplying $-b$ by a gives $-ab$, therefore if the sign of the multiplier is changed, of course the sign of the product must also be changed. As a proof, it is like saying that if A, a white man, wears black shoes, therefore it follows that B, a black man, must wear shoes of an opposite color.

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* Smith, D.E. *The Teaching of Elementary Mathematics,* Macmillan Co., 1900, p.188.

Smith also wrote a popular two volume history of mathematics.

ELEMENTS OF ALGEBRA,

BY

LEONARD EULER,

TRANSLATED FROM THE FRENCH;

WITH THE

NOTES OF M. BERNOULLI, &c.

AND THE

ADDITIONS OF M. DE LA GRANGE.

FOURTH EDITION,
CAREFULLY REVISED AND CORRECTED.

BY THE REV. JOHN HEWLETT, B.D. F.A.S. &c.

TO WHICH IS PREFIXED,

A Memoir of the Life and Character of Euler,

BY THE LATE

FRANCIS HORNER, ESQ., M. P

LONDON:
PRINTED FOR LONGMAN, REES, ORME, AND CO.
PATERNOSTER-ROW.

1828.

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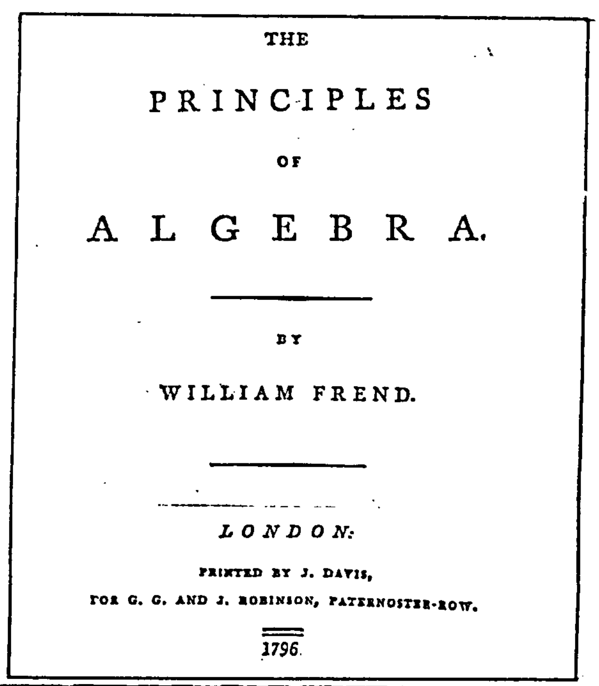
The Negative numbers

Worksheet: The last of the opposition - W. Frend

The following extracts are taken from *The Principles of Algebra* by William Frend (1757-1841), published in 1796.

Frend, an Englishman, was a rebel in more senses than one, and was banished from Cambridge where he had been a tutor*. He was also the father-in-law of a much more famous mathematician, Augustus De Morgan, whom we shall quote in the answer sheet.


THE

PRINCIPLES

OF

ALGEBRA.

BY

WILLIAM FREND.

LONDON:

PRINTED BY J. DAVIS,

FOR G. G. AND J. ROBINSON, PATERNOSTER-ROW.

1796.

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* For further details on Frend see, for example
De Morgan A., *A Budget of Paradoxes, Vol. 1*, Open Court, 1915.

I. ...The firſt error in teaching the principles of algebra is obvious on peruſing a few pages only in the firſt part of Maclaurin's Algebra. *Numbers are there divided into two ſorte, poſitive and negative; and an attempt is made to explain the nature of negative numbers, by alluſions to book-debts and other arts. Now, when a perſon cannot explain the principles of a ſcience without reference to metaphor, the probability is, that he has never thought accurately upon the ſubject. A number may be greater or leſs than another number; it may be added to, taken from, multiplied into, and divided by another number; but in other reſpects it is very untractable: though the whole world ſhould be deſtroyed, one will be one, and three will be three; and no art whatever can change their nature. You may put a mark before one, which it will obey: it ſubmits to be taken away from another number greater than itſelf, but to attempt to take it away from a number leſs than itſelf is ridiculous. Yet this is attempted by algebraiſts, who talk of a number leſs than nothing, of multiplying a negative number into a negative number and thus producing a poſitive number, of a number being imaginary.

...This is all jargon, at which common ſenſe recoils; but, from its having been once adopted, like many other figments, it finds the moſt ſtrenuous ſupporters among thoſe who love to take things upon truſt, and hate the labour of a ſerious thought.

II. ...It is improper to write thus: $-a + b \times c = \frac{d}{e}$; it ſhould be $b \times c - a = \frac{d}{e}$. Again, to write thus, $b \times c - a = -f$, is not only improper,

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* Colin Maclaurin (1698-1746), a student of Newton, made contributionsto Analysis. About Maclaurin's Algebra, among other "textbooks" published in the 18th century, see the answer sheet: Euler.

13

but abſurd, as will be ſeen by attempting to read it. From b into c take a, the remainder is equal to ſome number which we will call m; but the mark — before f denotes that it is to be taken away from ſome number which is not written down, and we cannot make any ſenſe of the expreſſion $-f$.

Again, we ſhould do worſe by writing down any of the above marks without any numbers. Thus $- \times - = +$ is as nonſenſical in algebra, as in common language to ſay, take away into take away equals add.

Questions

1. Summarise Frend's arguments against the use of negative numbers as expressed in the two extracts.

2. How would you answer him?

Having seen Frend's general attitude to the subject of negative numbers, we now look at his treatment of certain algebraic topics.

(100)

ON EQUATIONS OF THE SECOND ORDER.

AN equation having, in any term, an unknown quantity of the ſecond power, and no higher power in any other term, is called an equation of the ſecond order, and all equations of this order are reduceable by the preceding rules to the following forms.

1. $x^2 = b$
2. $x^2 + ax = b$
3. $x^2 - ax = b$
4. $ax - x^2 = b.$

Questions

3. a) Does Frend, in fact, include all possible equations of second order in his representation? (Remember that a and b represent positive numbers only.)

   b) In which of the four cases may Frend run into a negative root?

   c) What do you consider to be the disadvantages of this approach?

The following extract is an example of how Frend solves a second order equation.

(111)

Let $x + \sqrt{5x + 10} = 8$.

$$\therefore \sqrt{5x+10} = 8 - x$$
$$\therefore 5x + 10 = 64 - 16x + x^2$$
$$\therefore 5x + 16x - x^2 = 64 - 10$$
$$\therefore 21x - x^2 = 54 \qquad \text{(a)}$$
$$\therefore \frac{441}{4} - 21x + x^2 = \frac{441}{4} - 54 = \frac{441-216}{4} = \frac{225}{4} \qquad \text{(b)}$$
$$\therefore \sqrt{\frac{441}{4} - 21x + x^2} = \sqrt{\frac{225}{4}}$$
$$\frac{21}{2} - x = \frac{15}{2}$$
$$\therefore x = \frac{21}{2} - \frac{15}{2} = \frac{6}{2} = 3.$$

In this caſe the root $x - \frac{21}{2}$ cannot be uſed; for, by the conditions of the queſtion, x is leſs than eight.

13

Questions

4. a) Explain the step from (a) to (b).

   b) Why does Frend maintain that "x is less than eight"?

   c) Would you also reject the second root?

   d) If Frend had begun the question at the point (a), would his treatment have been the same?

In the previuos sections we have seen how Frend deals with equations of second order. In the following extract (taken from an appendix to Frend's book, written by Francis Maseres (1731-1824), an Englishman and member of the Royal Society) we see how the "negative roots" obtained by other mathematicians are explained. Note that we have deliberately ommited some bits of the text - see Qu. 5.

(466)

..... as to the *negative roots* of an equation, they are in truth the real and poſitive roots of another equation conſiſting of the ſame terms as the firſt equation, but with different ſigns + and — prefixed to ſome of them; ſo that, when writers of Algebra talk of the negative roots of an equation, they, in fact, jumble two different equations together, and ſuppoſe the propoſed, or firſt, equation to have not only its own proper roots (which they call its *affirmative*, or *poſitive*, roots,) but to have likewiſe the roots of a different equation, which they call its *negative* roots. Thus, for example, they would ſay, that the quadratick equation $xx + 4x = 320$, has two roots, to wit, the poſitive, or affirmative, root, , and the negative root, . But this latter number, , is, in truth, the root of a different equation, to wit, of the equation . So that this kind of abſurd and fantaſtick language only tends to the confounding together the two different equations $xx + 4x = 320$, and , and conſidering them as if they were one and the ſame equation.

Questions

5. a) Fill in the bits we have omitted in the text.

   b) Apply the above remarks to $x^2 - 5x = 50$ and $4x - x^2 = 3$.

   c) Does this extract remind you of previous work in this series of worksheets?

AN

APPENDIX,

BY

FRANCIS MASERES, Esq. F.R.S.

CURSITOR BARON OF HIS MAJESTY'S COURT OF EXCHEQUER.

Algebram verò, ut ſolet doceri, *animadverti certis regulis et numerandi formulis ita eſſe contentam, ut videatur potiùs Ars quædam confuſa, cujus uſu Ingenium quodam modo turbatur et obſcuratur, quàm Scientia, quâ excolatur et perſpicacius reddatur.*

Des Cartes in his Diſſertation *De Methodo,* page 11.

A. Arcavi
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1983

THE WEIZMANN INSTITUTE OF SCIENCE
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DEPARTMENT OF SCIENCE TEACHING

The Negative numbers

Answer sheet: The last of the opposition - W. Frend

1. Frend's opposition to the use of negative numbers is based on a number of points. In the first extract we have the following:

   i) The explanations of negative numbers uses examples from outside mathematics -

   > ...an attempt is made to explain the nature of negative numbers, by allufions to book-debts and other arts. Now, when a perfon cannot explain the principles of a fcience without reference to metaphor, the probability is, that he has never thought accurately upon the fubject.

   ii) A number is a number - its essential nature is not changed by having "a mark" put before it.

   iii) A number "will obey" a mark put before it - as long as it makes sense to do so. Thus

   > ...it fubmits to be taken away from another number greater than itfelf, but to attempt to take it away from a number lefs than itfelf is ridiculous.
   >
   > ...This is all jargon, at which common fenfe recoils

In the second extract we have the following points:

iv) The expression (-f) makes no sense, because

> the mark — before *f* denotes that it is to be taken away from fome number which is not written down,

v) The marks (symbols) without numbers make no sense. Thus - x - = + is nonsense both in algebra and in everyday language.

2. The following two extracts, written by two famous English mathematicians, who both were also skilled writers on mathematics, can be used as the basis for answering Frend.

The first extract is taken from Whitehead, A.N., *An Introduction to Mathematics,* first published in the Everyman series of texts in the early part of the century. Whitehead (1861-1947), with Bertrand Russell (1872-1970) made a considerable contribution to the foundations and philosophy of mathematics. The work in question is an extremely readable popularisation of elementary mathematics.

14

Consider the statement $2+3=5$. We add 3 to 2 and obtain 5. Think of the operation of adding 3: let this be denoted by $+3$. Again $4-3=1$. Think of the operation of subtracting 3: let this be denoted by -3. Thus instead of considering the real numbers in themselves, we consider the *operations* of adding or subtracting them: instead of $\sqrt{2}$, we consider $+\sqrt{2}$ and $-\sqrt{2}$, namely the operations of adding $\sqrt{2}$ and of subtracting $\sqrt{2}$. Then we can add these operations, of course in a different sense of addition to that in which we add numbers. The sum of two operations is the single operation which has the same effect as the two operations applied successively. In what order are the two operations to be applied? The answer is that it is indifferent, since for example

$$2+3+1=2+1+3;$$

so that the addition of the steps $+3$ and $+1$ is commutative.

Mathematicians have a habit, which is puzzling to those engaged in tracing out meanings, but is very convenient in practice, of using the same symbol in different though allied senses. The one essential requisite for a symbol in their eyes is that, whatever its possible varieties of meaning, the formal laws for its use shall always be the same. In accordance with this habit the addition of operations is denoted by $+$ as well as the addition of numbers. Accordingly we can write

$$(+3)+(+1)=+4;$$

where the middle $+$ on the left-hand side denotes the addition of the operations $+3$ and $+1$. But, furthermore, we need not be so very pedantic in our symbolism, except in the rare instances when we are directly tracing meanings; thus we always drop the first $+$ of a line and the brackets, and never write two $+$ signs running. So the above equation becomes

$$3+1=4. \quad \ldots$$

What is the use of all this elaboration? ...

The answer is that what the mathematician is seeking is Generality. ...

Any limitation whatsoever upon the generality of theorems, or of proofs, or of interpretation is abhorrent to the mathematical instinct. ...

Let us see how generality is gained by the introduction of this idea of operations. Take the equation $x+1=3$; the solution is $x=2$. Here we can interpret our symbols as mere numbers, and the recourse to "operations" is entirely unnecessary. But, if x is a mere number, the equation $x+3=1$ is nonsense. For x should be the number of things which remain when you have taken 3 things away from 1 thing; and no such procedure is possible. At this point our idea of algebraic form steps in, itself only generalization under another aspect. We consider, therefore, the general equation of the same form as $x+1=3$. This equation is $x+a=b$, and its solution is $x=b-a$. Here our difficulties become acute; for this form can only be used for the numerical interpretation so long as b is greater than a, and we cannot say without qualification that a and b may be any constants. In other words we have introduced a limitation on the variability of the "constants" a and b, which we must drag like a chain throughout all our reasoning. Really prolonged mathematical investigations would be impossible under such conditions. Every equation would at last be buried under a pile of limitations. But if we now interpret our symbols as "operations," all limitation vanishes like magic. The equation $x+1=3$ gives $x=+2$, the equation $x+3=1$ gives $x=-2$, the equation $x+a=b$ gives $x=b-a$ which is an operation of addition or subtraction as the case may be.

The second quotation is taken from De Morgan, A., *A Budget of Paradoxes,* Open Court, 1915. Augustus De Morgan died in 1871. The year of his birth can be found from the puzzle he himself proposed - "I was x years old in the year x^2". He studied at Cambridge, where he was tutored by George Peacock - who will be the subject of another worksheet in this series. Later, he became professor of mathematics at University College, London. De Morgan married Frend's daughter, which did not prevent him disagreeing with Frend's views on mathematics.
De Morgan was a prolific author, and his works are always interesting.

Augustus De Morgan

14

The *Budget of Paradoxes* is, as its name suggests, a collection of paradoxes, but the word does not have its usual meaning - De Morgan writes - *"I use the word in the old sense: a paradox is something which is apart from general opinion, either in subject matter, method, or conclusion."*

Frend is clearly a paradoxer under this definition - since, in his time, there were few serious opponents of negative numbers left.

The following is the quotation. It includes a passage from Maseres*, who also belonged to the "negative paradoxers", and De Morgan's response.

> Maseres speaks as follows: "A single quantity can never be marked with either of those signs, or considered as either affirmative or negative; for if any single quantity, as *b*, is marked either with the sign + or with the sign — without assigning some other quantity, as *a*, to which it is to be added, or from which it is to be subtracted, the mark will have no meaning or signification: thus if it be said that the square of —5, or the product of —5 into —5, is equal to +25, such an assertion must either signify no more than that 5 times 5 is equal to 25 without any regard to the signs, or it must be mere nonsense and unintelligible jargon.

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* Maseres, F., *A Dissertation on the use of the Negative Sign in Algebra ...*, London, 1758.

To which De Morgan replies:

> ...The great difficulty of the opponents of algebra lay in want of power or will to see extension of terms. ... One of my paradoxers was present at a meeting of the Royal Society (in 1864, I think) and asked permission to make some remarks upon a paper. He rambled into other things, and, naming me, said that I had written a book in which two sides of a triangle are pronounced *equal* to the third.[5] So they are, in the sense in which the word is used in complete algebra; in which A+B = C makes A, B, C, three sides of a triangle, and declares that going over A and B, one after the other, is equivalent, in change of place, to going over C at once. My critic, who might, if he pleased, have objected to extension, insisted upon reading me in unextended meaning.

[5] The paper was probably one on complex numbers, or possibly one on quaternions, in which direction as well as absolute value is involved.

The extracts emphasize the concept of extension and the search for generality in mathematics. They are fundamental in understanding mathematical activity in general, and in answering Frend in particular.

Specific replies to the points made by Frend may include the following:

i) The use of an example from outside mathematics, he describes as "The first error in teaching the principles of algebra". But it is just in the teaching of mathematics, that a physical or other example can be useful, either to motivate or to reinforce (cf. Saunderson's remark about the double negative in grammar). Clearly, mathematics demands something more rigorous in the end, but examples and models are useful didactical strategies.

14

Whitehead suggests a mathematical interpretation for -5 as the *operation* of subtracting 5, and then defines what he means by adding and subtracting these operations themselves, thus indulging in a number of "extensions of terms".
It is worth noting that it is not that easy to extend Whitehead's operations to multiplication of negative numbers.

ii) Frend's conception of number (as for many of his contemporaries) is that of a physical magnitude, and he is unwilling to extend. The search for extension and generality, as already remarked, are the essence of mathematical activity. Thus Frend would oppose much progress in mathematics.
The problem with answering Frend is that we are talking about undefined entities, and as long as we do not have an explicit definition of the negative numbers, we cannot argue with complete objectivity.

The mathematician, C.L. Dodgson (1832-1898), better known as Lewis Carroll, made Humpty Dumpty say:

> *When I use a word it means just what I choose it to mean - neither more nor less.**

Here, as elsewhere, Carroll's mathematical background is not far under the surface. A mathematician may define a term to mean what he wishes as long as it is consistent with previous definitions and results.

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* Carroll L., *The Annotated Alice, with Introduction and notes by Martin Gardner.* Meridian, 1960, p. 269.

The apparent inconsistencies of Wallis, etc. had to be dealt with, but Frend's objection is more a matter of prejudice than logic.

iii) If we extend the concept of number to include negative numbers, then we can also subtract a greater from a smaller. Frend's difficulty is caused, as in ii), by his refusal to extend the number concept.

iv) In the second extract, Frend's problems are compounded by his further unwillingness to allow the minus sign to have any other meaning than subtraction, and that only as in iii). In writing -f, we are giving the sign -, the meaning of "negative", rather than subtraction; or, as Whitehead interprets it, as an operation. Again, we need the extension principle, of attaching new or extended meaning to familiar symbols.
De Morgan's + example, is another illustration of this.

In this connection it is interesting to note that some authors and texts distinguish between the two roles of the minus and other signs. For example, instead of

$$(-4) - (-3)$$

they write $(^-4) - (^-3)$.

v) An abbreviation is admissible if it does not lead to misunderstanding. The use of " - x - = + " is merely as an abbreviation for the concept
14 "a negative number times a negative number gives

a positive number". We have already remarked on a possible didactical difficulty in the Euler answer sheet, but, in general, it is certainly acceptable as a convenient shorthand. Frend's objection was probably fired by over enthusiasm for his cause.

Although, the "true mathematical" explanation of negative numbers troubled many at the time of Frend, few serious opponents of negative numbers were left. The following quotation* illustrates this:

> *Although a study of British algebra textbooks and articles published in the leading British scientific journals of the late 18th and early 19th centuries demonstrates that the problem of negative numbers was a major concern of British thinkers of the period, it also shows that most British mathematicians were reluctant to renounce the riches of algebra because of foundational problems. Although admitting the absence of a satisfactory definition of the negatives, most argued in favor of their retention, if only because of practical considerations. Thinkers who concurred with Maseres and Frend in recognition of the problem but not in their solution included William Greenfield, an amateur mathematician and a professor of rhetoric at the University of Edinburgh,...*

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* Pycior H., George Peacock and the British Origins of Symbolical Algebra, *Historia Mathematica,* 1981, Vol. 8, p. 23-45.

Greenfield said*:

...The truth is, that the whole bufinefs of algebra might be carried on without the confideration of the negative roots. The difference between fuch a fyftem and the prefent, is precifely this; that wherever a problem required us to confider any of the quantities, as exifting in oppofite fituations; wherever, for inftance, a line or a point was to be confidered, as fituated firft on the right hand, and then on the left; it would be neceffary, to find and to refolve a feparate equation for each of thefe cafes. Thus, in the analyfis of any particular curve, it would be neceffary to have a feparate equation for each of the four angles of the co-ordinates; except, indeed, the axes were fo chofen, as to make us certain that there were fome of thefe angles, in which no part of the curve was to be found. Since, therefore, the ufe of negative quantities frees us from this inconvenience, which, in many cafes, particularly in the analyfis of curves, would be exceedingly perplexing; and fince it evidently affords fo great elegance and univerfality to algebraical folutions; to find our author gravely declaring that he can fee no advantage in it, is perfectly aftonifhing: As it is to be lamented, that he did not exert his induftry and ingenuity, rather to confirm than to deftroy; rather to demonftrate, how far we might rely on the method of negative quantities, than to overturn at once fo great a part of the labours of the modern algebraifts.

Summing up, this period in the history of negative numbers can be considered as an example of one of Wilder's**'laws' governing the evolution of mathematical concepts: *The admissibility and acceptance of a concept will be decided by its degree of fruitfulness. In particular, a concept will not be forever rejected because of its origin or on the grounds of metaphysical criteria such as "unreality".*

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* In his address to the *Royal Society of Edinburgh (Trans.)*, 1788, Vol. 1, p. 131-145.

** Wilder R.L., *Evolution of Mathematical Concepts*, Transworld Student Library, 1973, p. 197-200.

14

3. a) The equations given by Frend include those cases only in which at least one root is positive. He does not include the equation

$$x^2 + ax + b = 0 \quad (a, b > 0)$$

because it has two negative (or imaginary) roots.

b) In the first three of Frend's cases one may obtain a negative root.

c) The fact that Frend has to consider four cases, instead of the one

$$ax^2 + bx + c = 0$$

in which a, b and c are unrestricted (real) numbers, is a considerable disadvantage. He discusses the solution of each of the cases separately. The student has to fit any given problem to the appropriate one of the four cases, involving unnecessary technique and memorisation.

4. a) The step from (a) to (b) is the standard method of completing the square. It is worthy of note that Frend "subtracts (a)" from $\frac{441}{4}$, to obtain the complete square. In this step, he uses the principle

$$A - (B - C) = A - B + C .$$

This does not imply any recognition of negative numbers, since Frend is careful to stipulate that $A > B$ and $B > C$. This rule of subtraction was already known even by those who did not deal with negatives at all (see also the Viète work- and answer sheets).

We give below the full text, in which Frend discusses the general case, of which the given problem is an example.

4. To resolve the equations of the fourth form, the mode adopted in the two preceding forms fails; for by adding $\frac{a^2}{4}$ to $ax - x^2$, one side of an equation of the fourth form, a compound term $ax - x^2 + \frac{a^2}{4}$ is made, which is not similar to the second power of a two-fold term. Instead therefore of adding the second power of half the co-part of the unknown term to each side of the equation

equation, each is taken away from $\frac{a^2}{4}$, and one side becomes the second power of a two-fold term, and the other side has only known numbers.

$$\text{From } \frac{a^2}{4} = \frac{a^2}{4}$$
$$\text{take } ax - x^2 = b.$$
$$\therefore \frac{a^2}{4} - ax + x^2 = \frac{a^2}{4} - b.$$
$$\therefore \sqrt{\frac{a^2}{4} - ax + x^2} = \sqrt{\frac{a^2}{4} - b}$$
$$\therefore \frac{a}{2} - x = \sqrt{\frac{a^2}{4} - b}$$
$$\therefore x = \frac{a}{2} - \sqrt{\frac{a^2}{4} - b}.$$

The root of a compound term $\frac{a^2}{4} - ax + x^2$ is either $\frac{a}{2} - x$, or $x - \frac{a}{2}$, according as $\frac{a}{2}$ is greater or less than x; but in the equation $ax - x^2 = b$, x may be greater or less than $\frac{a}{2}$, and yet, in both cases, the difference between ax and x^2 be equal to some number: but x cannot be greater than a, for then x^2 could not be taken away from ax.

$ax - x^2 = x \times \overline{a - x}$; consequently x must be less than a, or $a - x$ will not represent any number whatever, since, if x is greater than a, it cannot be taken from a. But the learner must take care, in writing down such an equation,

(110)

that the fecond power of half the co-part of x is not lefs than b; for then $\sqrt{\frac{a^2}{4} - b}$ would be an abfurdity: b could not then be taken from $\frac{a^2}{4}$, and confequently $\sqrt{\frac{a^2}{4} - b}$ could not reprefent any number whatever.

$$\sqrt{x^2 - ax + \frac{a^2}{4}} = \sqrt{\frac{a^2}{4} - b}$$

$$\therefore x - \frac{a}{2} = \sqrt{\frac{a^2}{4} - b}$$

$$\text{and } \frac{a}{2} - x = \sqrt{\frac{a^2}{4} - b}$$

From the firft of the two latter equations,

$$x = \sqrt{\frac{a^2}{4} - b} + \frac{a}{2}.$$

From the fecond,

$$x = \frac{a}{2} - \sqrt{\frac{a^2}{4} - b}.$$

b) Given that Frend does not recognize negative quantities, the first equation

$$x + \sqrt{5x + 10} = 8$$

implies that both x and $\sqrt{5x + 10}$ are less than 8.

c) The second root is x = 18, which Frend rejects as unsuitable because of "the conditions of the question". But, quite independently of Frend's particular standpoint, 18 does not satisfy the original equation. It arises in the third line, where both sides of the equation in line two, have been squared. Not only

$$\sqrt{5x + 10} = 8 - x$$

leads to line 3, but also

$$-\sqrt{5x+10} = 8 - x$$

and 18, is the solution of the latter.

In the modern texts the $\sqrt{}$ symbol is usually taken to mean the positive root only, but some use it as equivalent to the exponent $\frac{1}{2}$, i.e. $\sqrt{x} = x^{\frac{1}{2}}$, in in which case, both roots of x are to be understood. In Frend, of course, no such ambiguity can arise. However, he might have made some capital out of the fact that the admission of negative numbers, even implicitly, could lead to ridiculous results such as $18 + 10 = 8$.

d) Starting from

$$21x - x^2 = 54 ,$$

he would procede as before, exemplifying the fourth of his cases, as brought at the end of the solution of part a). And this time both roots 3 and 18, are admissible, since they are both positive and each step in the argument involves at most subtraction of a smaller quantity from a greater one.

5. a) The "unexpurgated" text reads as follows:

as to the *negative roots* of an equation, they are in truth the real and poſitive roots of another equation conſiſting of the ſame terms as the firſt equation, but with different ſigns + and — prefixed to ſome of them; ſo that, when writers of Algebra talk of the negative roots of an equation, they, in fact, jumble two different equations together, and suppose the proposed, or first, equation to have not only its own proper roots (which they call its *affirmative*, or *positive*, roots,) but to have likewise the roots of a different equation, which they call its *negative* roots. Thus, for example, they would say, that the quadratick equation $xx + 4x = 320$, has two roots, to wit, the positive, or affirmative, root, $+ 16$, and the negative root, $- 20$. But this latter number, 20, is, in truth, the root of a different equation, to wit, of the equation $xx - 4x = 320$. So that this kind of absurd and fantastick language only tends to the confounding together the two different equations $xx + 4x = 320$, and $xx - 4x = 320$, and considering them as if they were one and the same equation.

b) The equation $x^2 - 5x = 50$ has two roots, 10 and -5. Maseres would claim that the -5 indicates that 5 is the root of another equation $x^2 + 5x = 50$, with the same terms but a change of sign.

There are no problems with $4x - x^2 = 3$, since it has two positive roots, 1 and 3.

c) Maseres is essentially using the result we met in the Descartes worksheet:

De plus il est aysé de faire en vne mesme Equation, que toutes les racines qui estoient fausses deuienent vrayes, & par mesme moyen que toutes celles qui estoiēt vrayes deuienent fausses : a sçauoir en changeant tous les signes + ou -- qui sont en la seconde, en la quatriesme, en la sixiesme, ou autres places qui se designent par les nombres pairs, sans changer ceux de la premiere, de la troisiesme, de la cinquiesme & semblables qui se designent par les nombres impairs.

Aaa 3

Post script

As Greenfield wrote in the extract quoted in answer to Qu. 2,

> it is to be lamented, that he did not exert his induſtry and ingenuity, rather to confirm than to deſtroy; rather to demonſtrate, how far we might rely on the method of negative quantities, than to overturn at once ſo great a part of the labours of the modern algebraiſts.

It is interesting to see how those who did try "rather to confirm than to destroy", went about the business. For example De Morgan*

> A father is fifty-six, and his son twenty-nine years old: when will the father be twice as old as the son? Let this happen x years from the present time; then the age of the father will be $56+x$, and that of the son $29+x$; and therefore, $56+x=2(29+x)=58+2x$, or $x=-2$. The result is absurd; nevertheless, if in the equation we change the sign of x throughout it becomes $56-x=58-2x$, or $x=2$. This equation is the one belonging to the problem: a father is 56 and his son 29 years old; when *was* the father twice as old as the son? the answer to which is, two years ago. In this case the negative sign arises from too great a limitation in the

* De Morgan, A., *On the study and difficulties of mathematics*, Open Court Pub. Co., 1898.

terms of the problem, which should have demanded how many years have elapsed or will elapse before the father is twice as old as his son?

Again, suppose the problem had been given in this last-mentioned way. In order to form an equation, it will be necessary either to suppose the event past or future. If of the two suppositions we choose the wrong one, this error will be pointed out by the negative form of the result. In this case the negative result will arise from a mistake in reducing the problem to an equation. In either case, however, the result may be interpreted, and a rational answer to the question may be given.

In this latter extract, De Morgan uses the same means as Frend, the change of sign in an equation, to change the sign of a root. But the treatment is qualitatively different.

We leave the final word to the philosopher Auguste Comte (1798-1857)*:

NEGATIVE QUANTITIES.

As to negative quantities, which have given rise to so many misplaced discussions, as irrational as useless, we must distinguish between their *abstract* signification and their *concrete* interpretation, which have been almost always confounded up to the present day. Under the first point of view, the theory of negative quantities can be established in a complete manner by a single algebraical consideration. The necessity of admitting such expressions is the same as for imaginary quantities, as above indicated; and their employment as an analytical artifice, to render the formulas more comprehensive, is a mechanism of calculation which cannot really give rise to any serious difficulty. ...

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* Comte, A., *The Philosophy of Mathematics.* Translated by Gillespie, N.Y. 1851, p.81.

It is far from being so, however, with their concrete theory. This consists essentially in that admirable property of the signs + and –, of representing analytically the oppositions of directions of which certain magnitudes are susceptible. This *general theorem* on the relation of the concrete to the abstract in mathematics is one of the most beautiful discoveries ...
It consists in this: if, in any equation whatever, expressing the relation of certain quantities which are susceptible of opposition of directions, one or more of those quantities come to be reckoned in a direction contrary to that which belonged to them when the equation was first established, it will not be necessary to form directly a new equation for this second state of the phenomena; it will suffice to change, in the first equation, the sign of each of the quantities which shall have changed its direction; and the equation, thus modified, will always rigorously coincide with that which we would have arrived at in recommencing to investigate ...

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M. Bruckheimer

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The Negative Numbers

Worksheet: George Peacock

In England, the death of Isaac Newton (1642-1727) was followed by a period of relatively low scientific activity in general, and in mathematics in particular. English mathematicians distanced themselves from other European mathematicians, in part because of the dispute between Newton and Leibniz (1646-1716) concerning precedence in the invention of the infinitesimal calculus. This chauvinism on the part of the English mathematicians, caused them to pay insufficient attention to mathematical advances on the continent. In addition, there were at that time men, like Frend and Masères, who wrote books against the use of negative numbers. The English mathematician George Peacock (1791-1858), together with others, began to work in order to "revive" English mathematics, and in particular, to put algebra on a sound scientific basis. To this end he wrote two books, *Arithmetical Algebra* (1842) and *Symbolical Algebra* (1845). In the distinction between these two types of algebra can be found Peacock's response to the arguments of Masères and Frend.*

Below we bring extracts from these books, in which Peacock explains the difference between arithmetical algebra and symbolical algebra and presents his *Principle of Permanence of Equivalent Forms*, by which he connects these two types of algebra.

* About Peacock and his contribution to mathematics one can find further details in A. Macfarlane, *Lectures on Ten British Mathematicians of the Nineteenth Century*, Wiley, 1916.

In arithmetical algebra, we consider symbols as representing numbers, and the operations to which they are submitted as included in the same definitions (whether expressed or understood) as in common arithmetic: the signs + and − denote the operations of addition and subtraction in their ordinary meaning only, and those operations are considered as impossible in all cases where the symbols subjected to them possess values which would render them so...

...thus in... expressions... like $a - b$, we must suppose a greater than b...

all results whatsoever, including negative quantities, which are not strictly deducible as legitimate conclusions from the definitions of the several operations, must be rejected as impossible, or as foreign to the science. ...

Symbolical algebra adopts the rules of arithmetical algebra, but removes altogether their restrictions: thus symbolical subtraction differs from the same operation in arithmetical algebra in being possible for all relations of value of the symbols or expressions...

It is this adoption of the rules of the operations of arithmetical algebra as the rules for performing the operations which *bear the same names* in symbolical algebra, which secures the absolute identity of the results in the two sciences as far as they exist in common: or in other words, all the results of arithmetical algebra which are deduced by the application of its rules, and which are general in form, though particular in value, are results likewise of symbolical algebra, where they are general in value as well as in form: thus the product of a^m and a^n, which is a^{m+n} when m and n are whole numbers, and therefore general in form though particular in value, will be their product likewise when m and n are general in value as well as in form...

This principle, in my former Treatise on Algebra, I denominated the "*principle of the permanence of equivalent forms*," and it may be considered as merely expressing the general law of transition from the results of arithmetical to those of symbolical algebra...

Upon this view of the principles of symbolical algebra, it will follow that its operations are determined by the definitions of arithmetical algebra, as far as they proceed in common, and by the "principle of the permanence of equivalent forms" in all other cases...

15

The results therefore of symbolical algebra, which are not common to arithmetical algebra, are generalizations of form, and not necessary consequences of the definitions, which are totally inapplicable in such cases. It is quite true indeed that writers on algebra have not hitherto remarked the character of the transition from one class of results to the other, and have treated them both as equally consequences of the fundamental definitions of arithmetic or arithmetical algebra: and we are consequently presented with forms of demonstration, which though really applicable to specific values of the symbols only, are tacitly extended to all values whatsoever...

The definition of a *power*, in Arithmetical Algebra, implies that its index is a whole number: and if this condition be not fulfilled, the definition has no meaning, and therefore no conclusions are deducible from it: the principles, however, of Symbolical Algebra, will enable us, not merely to recognise the existence of such powers, but likewise to give, in many instances, a consistent interpretation of their meaning.

Questions

1. What is the difference between the expression $a - b$ in arithmetical algebra and in symbolical algebra?

2. What is the difference between what Peacock calls arithmetic and what we call arithmetic today?

3. In view of Peacock's definition of arithmetical algebra, symbolical algebra and the *Principle of Permanence of Equivalent Forms*, find where the principle is used in each of the following and if this use is justified.

   i) We know that

   $$a^m \cdot a^n = a^{m+n}, \qquad m, n \text{ natural numbers.}$$

   Substitute $n = 0$ to obtain

   $$a^m \cdot a^0 = a^{m+0} = a^m,$$

   Whence $a^m \cdot a^0 = a^m$.

   Therefore $a^0 = 1$.

   ii) We know that

   $$(a-b)\cdot(c-d) = a\cdot c - b\cdot c - a\cdot d + b\cdot d,$$

   when $c > d > 0$, $a > b > 0$.

   Substitute $a = 0$, $c = 0$, to obtain

   $$(-b)\cdot(-d) = +b\cdot d .$$

   iii) If a and b represent natural numbers, the distance of $a + b$ from 0 on the number line, is equal to the distance of a from 0 plus the distance of b from 0. We conclude that for every a, b the distance of $a + b$ from 0 is equal to the distance of a from 0 plus the distance of b from 0.

15

iv) If a, b and c are natural numbers and

$$a > b$$

then $$ac > bc.$$

We conclude that for every a, b and c

$$a > b \Rightarrow ac > bc.$$

v) For a, b natural numbers and $a > b$

$$(a^2 - b^2)/(a - b) = a + b.$$

We conclude that for every a and b

$$(a^2 - b^2)/(a - b) = a + b.$$

vi) For a, m natural numbers

$$a^m = \underbrace{a \cdot a \ldots \cdot a}_{m \text{ times}}$$

Substitute $a = 0$,

whence $0^m = 0$.

Substitute $m = 0$,

whence $0^0 = 0$.

4. The following is an extract from the textbook *First Course in Algebra, Part I (SMSG, Yale Univ. Press, 1961, p. 145-6)*. Discuss the approach in the light of Peacock.

Chapter 7

PROPERTIES OF MULTIPLICATION

7-1. Multiplication of Real Numbers

Now let us decide how we should multiply two real numbers to obtain another real number. All that we can say at present is that we know how to multiply two non-negative numbers.

Of primary importance here, as in the definition of addition, is that we maintain the "structure" of the number system. We know that if a, b, c are any numbers of arithmetic, then

$$ab = ba,$$
$$(ab)c = a(bc),$$
$$a \cdot 1 = a,$$
$$a \cdot 0 = 0,$$
$$a(b + c) = ab + ac.$$

(What names did we give to these properties of multiplication?) Whatever meaning we give to the product of two real numbers, we must be sure that it agrees with the products which we already hav· for non-negative real numbers and that the above properties of multiplication still hold for all real numbers.

Consider some possible products:

$(2)(3)$, $(3)(0)$, $(0)(0)$, $(-3)(0)$, $(3)(-2)$, $(-2)(-3)$.

(Do these include examples of every case of multiplication of pos: tive and negative numbers and zero?) Notice that the first thre· products involve only non-negative numbers and are therefore already determined:

$$(2)(3) = 6, \quad (3)(0) = 0, \quad (0)(0) = (0).$$

Now let us try to see what the remaining three products will have to be in order to preserve the basic properties of multiplication listed above. In the first place, if we want the multiplication property of 0 to hold for all real numbers, then we must have $(-3)(0) = 0$. The other two products can be obtained as follows:

$0 = (3)(0)$	
$0 = (3)\big(2 + (-2)\big)$,	by writing $0 = 2 + (-2)$; (Notice how this introduces a negative number into the discussion.)
$0 = (3)(2) + (3)(-2)$,	if the distributive property is to hold for real numbers;
$0 = 6 + (3)(-2)$,	since $(3)(2) = 6$.

We know from uniqueness of the additive inverse that the only real number which yields 0 when added to 6 is the number -6. Therefore, if the properties of numbers are expected to hold, the only possible value for $(3)(-2)$ which we can accept is -6.

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The Negative Numbers

Answer sheet: George Peacock

1. Peacock, in his *Symbolical Algebra*, explains it as follows:

> ... in the expression $a-b$, if we are authorized to assume a to be either *greater* or *less* than b, we may replace a by the equivalent expression $b+c$ in one case, and by $b-c$ in the other: in the first case, we get $a-b=b+c-b=b-b+c$ (Art. 22), $=0+c$ (Art. 16) $=+c=c$: and in the second $a-b=b-c-b$ $=b-b-c=0-c=-c$. The first result is recognized in Arithmetical Algebra (Art. 23): *but there is no result in Arithmetical Algebra which corresponds to the second:* inasmuch as it is assumed that no operation can be performed and therefore no result can be obtained, when a is less than b, in the expression $a-b$ (Art. 13).*
>
> 545. Symbols, preceded by the signs + or −, without any connection with other symbols, are called *positive* and *negative* (Art. 32) symbols, or *positive* and *negative* quantities: such symbols are also said to be *affected* with the signs + and −. *Positive* symbols and the numbers which they represent, form the subjects of the operations both of Arithmetical and Symbolical Algebra: but *negative* symbols, whatever be the nature of the quantities which the *unaffected* symbols represent, belong exclusively to the province of Symbolical Algebra.
>
> * If we assume symbols to be capable of all values, from *zero* upwards, we may likewise include *zero* in their number: upon this assumption, the expressions $a+b$ and $a-b$ will become $0+b$ and $0-b$, or $+b$ and $-b$, or b and $-b$ respectively, when a becomes equal to *zero*: this is another mode of deriving the conclusion in the text.

2. Peacock, for example, does not regard the four operations with negative numbers as arithmetic, since the negative quantity does not have any place whatsoever in arithmetic, as we have seen in the previous question. And this is common to many mathematicians in the 19th century.

Nowadays, the word arithmetic has a much more wide meaning. We read in Lapede's *Dictionary of Physics and Mathematics* (McGraw-Hill, 1978) the following definition of arithmetic:

> *Addition, subtraction, multiplication and division usually of integers, rational numbers, real numbers, or complex numbers.*

Very often, operations between other mathematical entities (such as vectors, matrices, etc.) are also called arithmetic.

3. In each of the sections we have made use of the *Principle of Permanence of Equivalent Forms.* What we have to analyze in each case, is whether this use is legitimate or leads us to contradictions.

   i) When we substituted $n = 0$ in $a^m a^n = a^{m+n}$, we used the principle; that is, we extended the application of the equality from the domain of arithmetical algebra to that of symbolical algebra.
   In this case, its use is justified since it does not lead us to contradictions.

   ii) When we substitute $a = 0$ *and* $c = 0$, we use the principle.
   Again this is justified.

In these two sections, the principle does not lead us astray, but on the other hand, we should be explicitly aware of the "logical status" of what we have done.
In the following, we bring two extracts related to these two sections, which warn against regarding the arguments based on the principle as proofs. They are not proofs, unless we accept the principle as an axiom, which we

cannot do quite so easily since, as we shall see in subsequent sections, it can also lead to incorrect and contradictory conclusions.

The first extract is taken from S.I. Brown, Signed Numbers: a "Product" of misconceptions. *The Mathematics Teacher,* 1969, 62, p. 183-195.

Preservation principle

The preservation principle is that if we wish to extend a concept in mathematics beyond its original definition then that candidate ought to be chosen which leaves as many principles of the old system intact as possible. The fact that these principles are left intact does not at all *prove* that a definition for elements of an extended set is correct. It merely motivates us to make a particular definition.

...Suppose we define a^n for n a natural number to be

$$a \cdot a \cdot \ldots \cdot a,$$

where a appears n times as a factor. For m, n, belonging to the set of natural numbers, it is easy to prove that

$$a^m \cdot a^n = a^{m+n}.$$

Now what is a^0? It obviously has no meaning in terms of the definition of a^n, since it is meaningless to select the number a zero times as a factor. "Proofs" are often offered...

The difficulty with the above "proof" obviously lies with the assertion that

$$a^0 \cdot a^n = a^{0+n}.$$

Why should we assume that 0 behaves (as an exponent) the same way the natural numbers do? The answer is that we cannot, since the assertion

$$a^n \cdot a^m = a^{n+m}$$

was proven only for natural numbers. There would be nothing inconsistent about our system if we maintained the above equality only for natural numbers and stipulated that it failed in the case of $n = 0$. We could then define a^0 in any way we wished and still have a consistent system. If, however, we wish the principle for multiplication to be preserved in the *new* system (which includes 0), then we are forced to *define* a^0 to be 1 as the "proof" suggests.

The second extract comes from F. Klein, *Elementary Mathematics From an Advanced Standpoint. Arithmetic, Algebra, Analysis.* Dover Publications, n.d. p.23-28 (originally published in German in 1908.)

...If we now look critically at the way in which negative numbers are presented in the schools, we find frequently the error of trying to prove the logical necessity of the rule of signs, corresponding to the above noted efforts of the older mathematicians. One is to derive $(-b)(-d) = +bd$ heuristically, from the formula $(a - b)(c - d)$ and to think that one has a proof, completely ignoring the fact that the validity of this formula depends on the inequalities $a > b, c > d$[1]. Thus the proof is fraudulent, and the psychological consideration which would lead us to the rule by way of the principle of permanence is lost in favor of quasi-logical considerations.

Of course the pupil, to whom it is thus presented for the first time, cannot possibly comprehend it, but in the end he must nevertheless believe it ...

In opposition to this practice, I should like to urge you, in general, never to attempt to make impossible proofs appear valid. One should convince the pupil by simple examples, or, if possible, let him find out for himself that, in view of the actual situation, *precisely these conventions, suggested by the principle of permanence, are appropriate in that they yield a uniformly convenient algorithm, whereas every other convention would always compel the consideration of numerous special cases.* To be sure, one must not be precipitate, but must allow the pupil time for the revolution in his thinking which this knowledge will provoke. And while it is easy to understand that other conventions are not advantageous, one must emphasize to the pupil how really wonderful the fact is that a general useful convention really *exists*; it should become clear to him that this is by no means self-evident.

iii) This statement in mathematical form is

$$|a + b| = |a| + |b|$$

and this is true if a and b are natural numbers.

If we use the principle to extend the result to the non-negative numbers, we have made valid use of the principle, since it does not lead us to contradictions.

But if we use the principle to extend the result to negative numbers, we obtain a contradiction, as is clear from the following example,

$$|-3 + 5| = |-3| + |5|$$
$$2 = 3 + 5$$
$$2 = 8 .$$

iv) If we extend the result $a > b \Rightarrow ac > bc$, from the natural numbers to all the positive numbers, all is well. But, if we include zero, then $c = 0$ leads to

$$a > b \Rightarrow 0 > 0 ,$$

which contradicts the elementary meaning of >, and the concept of order which it is meant to express.

So, at least, we would require the exclusion of $c = 0$.

But worse is to come if we try to extend to the negative numbers. Suppose $a \neq 0$ and let $b = -a$, that is our result becomes

$$a > -a \Rightarrow ac > -ac$$

Now substitute $c = -1$ and we obtain

$$a > -a \Rightarrow a(-1) > -a(-1) ,$$

i.e. $$a > -a \Rightarrow -a > a .$$

which is completely contradictory to the basic property of order; that is, for any two numbers a, b, *just one* of the following three possibilities is correct,

$$a = b \ , \quad a > b \ , \quad a < b \ .$$

v) For natural numbers (where the subtraction $a - b$ is possible if and only if $a > b$), we can assert that

$$\frac{a^2 - b^2}{a - b} = a + b \qquad (1)$$

since $a^2 - b^2 = (a + b)(a - b)$ and we may cancel the factor $a - b$.

If we extend the result from the natural numbers to the integers (and also to the real numbers), all is well except for the case when $a = b$.

If, for example, we substitute in (1) $a = b = \frac{1}{2}$, we have

$$\frac{0}{0} = 1$$

but if we substitute $a = b = 3$ we have

$$\frac{0}{0} = 6 \; .$$

In general, we have $\frac{0}{0} = 2a$. Thus the extension is invalid.

vi) In this case, we used Peacock's Principle twice. First, we extend a^m to include $a = 0$, whence $0^m = 0$. And all is well.

Then, we used the Principle again to extend to $m = 0$, obtaining $0^0 = 0$.
But another possible extension can be obtained from section i) of this question, where we found that $a^0 = 1$ for all a, also by means of the Principle - whence by a second extension, we have $0^0 = 1$.

So, we are in a similar situation to the previous section. In fact, both $\frac{0}{0}$, and 0^0 are indeterminate expressions.

In conclusion, the *Principle of Permanence of Equivalent Forms* can be used to suggest the extension of concepts, but it is not a universal principle and one has to use it with care. If we look back, at previous worksheets, we can find hints of its implicit use.

- Saunderson notices that if we multiply the terms of an arithmetic series (of natural numbers) by a natural number, then the result will be also an arithmetic series. This property he extends to negative numbers and thus makes a justified use of the principle.

- Arnauld tries to use the principle on the ratio concept and fails, because in this case it is unjustified, as we saw in section iv).

- Bell in *The Development of Mathematics* discusses Peacock and his principle and sees in it a further weakness.

> It is difficult to see what the principle means, or what possible value it could have even as a heuristic guide. If taken at what appears to be its face value, it would seem to forbid $ab = -ba$, one of the most suggestive breaches of elementary mathematical etiquette ever imagined, as every student of physics knows from his vector analysis. As a parting tribute to the discredited principle of permanence, we note that since $2 \times 3 = 3 \times 2$, it follows at once from the principle that $\sqrt{2} \times \sqrt{3} = \sqrt{3} \times \sqrt{2}$. But the necessity for proving such simple statements as the last was one of the spurs that induced Dedekind in the 1870's to create his theory of the real number system. According to that peerless extender of the natural numbers, "Whatever is provable, should not be believed in science without proof."

And B. Russell* wrote:

> The possibility of a deductive Universal Algebra is often based upon a supposed principle of the Permanence of Form. Thus it is said, for example, that complex numbers must, in virtue of this principle, obey the same laws of addition and multiplication as real numbers obey. But as a matter of fact there is no such principle. In Universal Algebra, our symbols of operation, such as + and ×, are variables, the hypothesis of any one Algebra being that these symbols obey certain prescribed rules. In order that such an Algebra should be important, it is necessary that there should be at least one instance in which the suggested rules of operation are verified. But even this restriction does not enable us to make any general formal statement as to all possible rules of operation. The principle of the Permanence of Form, therefore, must be regarded as simply a mistake: other operations than arithmetical addition may have some or all of its formal properties, but operations can easily be suggested which lack some or all of these properties.

From the philosophical point of view there is no reason why the principle should hold unless we want it to. But from the didactical point of view, as we saw in the previous sections, is often extremely useful.

4. The approach in the SMSG textbook could be considered as a conceptual inheritance from Peacock. In fact, the paragraph can be seen to be based on a critical use of Peacock's Principle. It is explicitly assumed that only a *specified* set of laws (commutative, associative, distributive, multiplication by zero) can be extended from the positive to the negative numbers and then become the axioms for the proofs which follow.

B. Russell, *The Principles of Mathematics,* Allen & Unwin Ltd., 1951, p. 377.

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The Negative Numbers

Worksheet: The formal entry of the negative numbers into mathematics

Part I: Definitions

In the long history of the negative numbers, there are periods in which advances were made, and periods of stagnation. But there are almost no "discoveries" on which one can put a name or a date. The first half of the nineteenth century saw a number of attempts to "legitimize" the various sets of numbers - negative numbers, real numbers and complex numbers - of which we have already seen one such attempt in the work of Peacock. Another attempt, which had practically no influence on subsequent work in this field, was that of Sir William Rowan Hamilton (1805-1865), in a paper entitled Theory of conjugate functions, or algebraic couples; with a preliminary and elementary essay on algebra as the science of pure time, published in 1837.* This paper "was an attempt to give a system of axioms or principles for analysis (for Hamilton, the Science of Algebra included analysis) so that it can be *a Science properly so called; strict, pure and independent; deduced by valid reasonings from its own intuitive principles; and thus not less an object of priori contemplation than Geometry...* . Hamilton believed that our

* In the Trans. Roy. Irish Academy, Vol. 17, 1837, p. 293-422.

intuitive notion of order in time is more deep-seated than that of order in space and, as geometry is founded on the latter, so can analysis be founded on the former. In his essay Hamilton attempts to use these principles to construct and explain positive and contra-positive (negative) numbers, fractional quantities (rational numbers), and the set of all real numbers, all towards showing that analysis, like geometry, can be a Science."* One of the reasons for Hamilton's lack of influence is"... besides a general dullness and heaviness of style, there is too much obscurity...", as he himself admits. But this paper contains many of the ideas used by other mathematicians later in the century, with more success. Hamilton's paper is too involved to use as a source in this story - and this is true of later sources of which we are aware. So, for the last chapter in our history of the negative numbers, we change the style of worksheet. As we have seen in previous worksheets, many problems arose as a result of the attempt to solve and explain the solution of equations (Frend, De Morgan, Descartes). In the spirit of history, we shall take the final step using a very simple equation, to construct the integers from the natural numbers.

At the end, we shall indicate how the complete set of negative numbers might be constructed from the natural numbers.

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* Mathews J., William Rowan Hamilton's Paper, 1837 on the Arithmetization of Analysis. *Archive for the History of the Exact Sciences,* 1978, Vol. 19, p. 177-200.

17

All that we have at our disposal to start with is the set of natural numbers, $\{1, 2, 3, \ldots\}$, and the operations on them which produce a natural number as outcome - i.e. the arithmetical algebra of Peacock.

Consider the equation

$$a + x = b\ , \qquad a, b \in N$$

where N is the set of natural numbers.

1. Under what conditions is it also true that $x \in N$? (In other words, when does the equation have a natural number solution?)

2. Given two equations with the *same* solution $x \in N$,

$$a + x = b$$

$$\text{and} \quad a_1 + x = b_1,$$

what is the relation between a, b and a_1, b_1 ?

3. The solution of $a + x = b$, if it exists, is determined uniquely by a and b. Hence, we may denote the solution by the ordered pair (a, b).

   i) To which natural number does the pair (a, b) correspond?

   ii) Find *all* the ordered pairs corresponding to the natural number n. All these ordered pairs are said to be *equivalent*.

4. Draw the graph of all the pairs corresponding to the natural numbers 1, 2, etc.

5. We know how to add natural numbers - and we have seen how we may represent a natural number by an ordered pair. Hence we can "add" the ordered pairs corresponding to natural numbers.

i) Define the addition of such pairs, and examine it for validity.

ii) Can you "add" ordered pairs without restriction, i.e. without leaving N and its arithmetic?

iii) Give the addition of pairs a graphical interpretation.

6. i) Define the "subtraction" of ordered pairs (which represent national numbers) and examine it for validity.

ii) Can you "subtract" ordered pairs at will?

iii) Give subtraction of pairs a graphical representation.

7. Consider again the original equation with a natural number solution

$$a + x = b , \qquad a, b \varepsilon N .$$

Write it in terms of ordered pairs.

8. In section 3 we saw that not all ordered pairs correspond to natural numbers. Which do not?

Now let's generalise. Let (a, b) be *any* ordered pair of natural numbers. Generalise the concept of *equivalent* pairs using *addition* of natural numbers only.

Why do you think we suggested using addition only, and not subtraction?

9. i) Return to sections 4 - 7 and generalise them in the light of the generalisation in section 8.

ii) In what sense is it now true to say that every equation of the form

17

$$a + x = b\ , \qquad a, b \in N\ ,$$

has a solution?

10. The extension achieved in sections 8 and 9 might be regarded as a "didactical heritage" of Peacock's law of permanence of equivalent forms.

    In what sense is it similar, and in what *different* ?

11. In our new system of all ordered pairs of natural numbers, identify the set of equivalent pairs which from now on we shall denote by

$$n\ , 0\ , -n \qquad\qquad n \in N$$

respectively.

A set of equivalent pairs we shall call an *integer,* and the set of integers is denoted by Z.

12. Examine the following statements.

    i) For all $z_1\ , z_2 \in Z$,

$$z_1 + z_2 \in Z$$

and $$z_1 - z_2 \in Z\ .$$

ii) Examine the following statement

For all $z_1, z_2 \in Z$

$$z_1 + x = z_2$$

has a solution $x \in Z$.

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Answer Sheet: The formal entry of the negative numbers into mathematics

Part I: Definitions

1. The equation $a + x = b$, $a, b \in N$, has a solution belonging to N if and only if $b > a$.

2. The two equations

$$a + x = b$$

$$a_1 + x = b_1$$

have the same solution $x \in N$, if and only if

$$b - a = b_1 - a_1 \quad \text{and} \quad b > a , \quad b_1 > a_1 .$$

The first half of this condition is equivalent to

$$b + a_1 = b_1 + a ,$$

to which we shall have cause to refer later.

3. i) The ordered pair (a, b) corresponds to the natural number $b - a$.

   ii) All the ordered pairs corresponding to $n \in N$ satisfy

$$b - a = n \qquad b > a , \quad a, b \in N .$$

   That is $b = a + n$.

   Hence all the pairs of the form $(a, a + n)$, $a \in N$ correspond to n.

For example, (8, 10), (2, 4), (100, 102) etc., all correspond to the number 2. All the pairs which represent a given natural number, are called *equivalent* pairs, and we denote this by the symbol $\sim$, for example:

$$(2, 4) \sim (8, 10).$$

<u>Three Methodological Notes</u>

I. A *number* in mathematics is an abstract concept which is represented in many ways. The simplest of these is the single *numeral*; for example 3 and III are numeral representations of the number concept three. Another representation is 11_2 (in base 2). In the process of doing mathematics, we represent numbers in many forms - thus 5 + 3 - 6 + 1 is also a representation of the number 3. In the context in which we are working at the moment, we are looking at certain particular representations of the natural numbers. The first is the representation in the form $b - a$, $b, a \in N$. This form is the natural one in the context, but it has one deficiency, the condition $b > a$, without which the expression $b - a$ does not make any sense. Therefore, we replaced this by another form of representation (a, b), for which it is still true that it represents a natural number if and only if $b > a$, but the ordered pair is still a legitimate symbol without this condition, since we have got rid of the operation which is undefined for $b < a$. The advantage of this apparently trivial technical step is that we now have a notation which generalises without difficulty, and as we see in the following, we thus obtain the integers. It should be remembered that all we are doing throughout
18 is formalising intuition, and although the use of $b - a$,

$b < a$, is formally illegitimate, intuitively it is entirely legitimate and almost essential as a guide to the formal steps.

II. There is another point, normally intuitively implicit in much practical mathematical work, which we need to make explicit in our present formal context. We first illustrate it by familiar example and then return to our context.

$$\frac{1}{2}, \frac{2}{4}, \frac{35}{70}, \frac{251}{502}, \ldots$$

are all representations of the same rational number. (There are, of course, many other representations of other forms, e.g. 0.5, but we are considering the one form only.) Sometimes a distinction is made between a *fraction,* which is one of these representations, and the *rational number* which is in some way the whole set of representations - the concept.

When we write

$$\frac{2}{4} = \frac{4}{8}$$

we are saying that $\frac{2}{4}$ and $\frac{4}{8}$ are two representations of the same number. The "fraction in its lower terms" is one way of choosing a particular representative from the set of representations. Given any two fractions, $\frac{a}{b}$ and $\frac{c}{d}$, they represent the same number if and only if

$$ad = bc .$$

Now let us rewrite this in formal mathematical language. Consider the set of all positive fractions $\frac{a}{b}$: $a, b \in N$. We say that two fractions $\frac{a}{b}$, $\frac{c}{d}$ are *equivalent* if and only if

$$ad = bc \ .$$

A subset (class) of equivalent fractions is called a rational number. Thus a rational number is a set of equivalent fractions. It is clear that we can remove the fraction from the development, and base our definition on the natural numbers only. All we have to do is to replace $\frac{a}{b}$ by the ordered pair (a, b). The condition for equivalence of such pairs remains the same since it is expressed entirely within N and operations on elements of N.

If we now define the arithmetic operations on the pairs (a, b), taking care to use definitions entirely valid within N, we have created the arithmetic of the positive rational numbers by formal mathematical definition from the arithmetic of the natural numbers.

This is precisely the logical skeleton of the method we are using to develop the arithmetic of the integers from the natural numbers.

The equation $a + x = b$ and the expression $b - a$ are the intuitive and motivational props; the ordered pair (a, b) is the basic element in our logical structure. The main point in the first three questions of this worksheet is the fact, that corresponding to any natural number, there are many representations in the form $b - a$, and hence in the form (a, b). So we introduce the idea of the set of all representations of a natural number, i.e. the idea of equivalent representations:

$$(a, b) \sim (c, d) \quad \text{if and only if } b - a = d - c$$

However there is a technical problem with this condition, which is the necessity for $b > a$, and it is precisely this condition which we wish to relax in order to build the set of integers. So we recast the equivalence in the form

$$(a, b) \sim (c, d) \text{ if and only if } b + c = a + d$$

which makes sense for all $a, b, c, d \in N$.

Now we are ready to use our intuition and to start defining the arithmetic of these ordered pairs and thus obtain the integers from the natural numbers on a rigorous mathematical basis.

III. Involved in the above argument is the concept of equivalence, which is a general mathematical concept of some importance. It is the mathematical formalisation of the physical sorting process. A proper unambiguous sorting of a set of objects requires that

i) every object is sorted;

ii) the categories into which the objects are sorted is not dependent on the order of sorting;

iii) an object is assigned to one and only one category.

Thus a collection of coloured plastic geometric shapes can be properly sorted according to shape or colour (or both), according to area, etc., where by the latter we mean that two shapes with the same area belong to the same category. On the other hand, if we instruct someone to sort the shapes placing in the same category any shape which has area greater than or equal to the shapes already there, he will fail to sort the shapes either altogether, or at least uniquely.

The mathematical formulation of the sorting process can be expressed as follows. Given a set S of elements a, b, c, ..., then a relation (denoted by $\sim$) between the elements of the set is an *equivalence relation* if for all, a, b and c $\in$ S

i) $a \sim a$, (reflexive property)

ii) $a \sim b \implies b \sim a$, (symmetric property)

iii) $a \sim b,\ b \sim c \implies a \sim c$, (transitive property).

(Compare with the three conditions of sorting above.) It can be proved that if these conditions are satisfied, then S can be sorted by $\sim$ into distinct *equivalence classes*.

In the above discussion we gave the example of fractions. Two fractions $\frac{m_1}{m_2}$ and $\frac{n_1}{n_2}$ are equivalent if and only if $m_1 n_2 = m_2 n_1$. Show that this relation on the set of fractions satisfies the three properties.

To return to our context: two ordered pairs of natural numbers (a, b) and (c, d), $b > a$ and $d > c$ are equivalent if and only if

$$\left.\begin{aligned} b &= a + n \\ d &= c + n \end{aligned}\right\} \text{ for some } n \in N,$$

$$\text{or} \quad a + d = b + c .$$

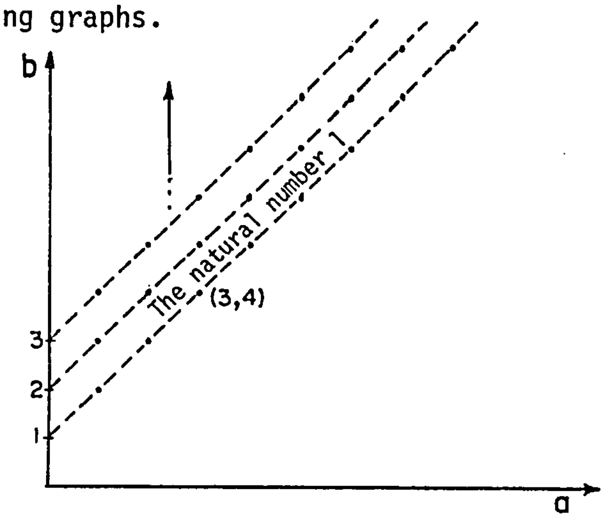
Show that this relation on the set of ordered pairs satisfies the three properties.

18

Note that we can drop the conditions b > a and so obtain an equivalence relation on the set of *all* ordered pairs (a, b), a, b ε N. See answer to Qu. 7.

<u>Continuation of solutions</u>

4. According to the choice of axes, we obtain one of the two following graphs.



Note that an equivalence class is represented by points all lying on the same straight line. The various lines do not intersect. To what general property of equivalence relations does this correspond?

5. i) $(a, b) \longrightarrow b - a \; \varepsilon \; N$

$(c, d) \longrightarrow d - c \; \varepsilon \; N$

Hence $(a, b) + (c, d) \longrightarrow b - a + d - c$

$= b + d - (a + c) \; \varepsilon \; N$

Thus we *define* $(a, b) + (c, d) = (a + c, b + d)$
The validity of this definition depends on two things - the first is that our definition is entirely within the arithmetic of N, which is

obviously the case - the second is that it is independent of the pairs we chose to represent the natural numbers. That is, if we choose any other representatives, say $(a_1, b_1) \sim (a, b)$ and $(c_1, d_1) \sim (c, d)$ then $(a_1 + c_1, b_1 + d_1) \sim (a + c, b + d)$.

The verification of this is a simple technical exercise. By the definition of equivalent pairs we have

$$b_1 - a_1 = b - a$$

$$d_1 - c_1 = d - c .$$

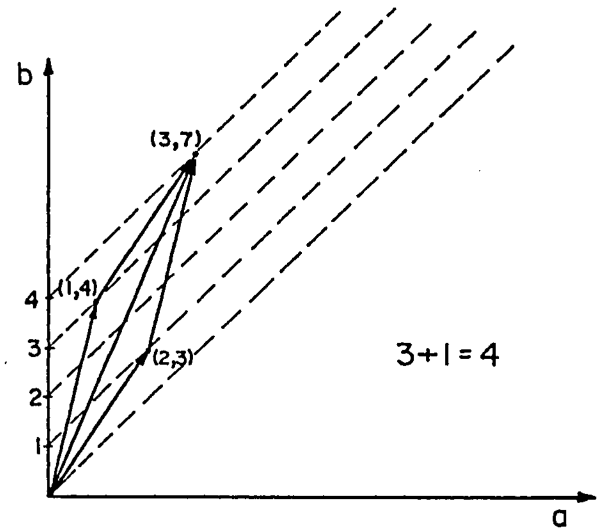
Hence $\quad b_1 + d_1 - (a_1 + c_1) = b + d - (a + c)$

i.e. $\quad (a_1 + c_1, b_1 + d_1) \sim (a + c , b + d)$.

ii) We have no restriction on the addition of ordered pairs, since a + c and b + d always belong to N.

iii) In order to perform the addition on the graph, we choose any two representatives of the numbers to be added, i.e. two points on the lines which represent the numbers. We draw the arrows from the origin to these two points to obtain two adjacent sides of a parallelogram, which we complete by drawing the other two sides. The fourth vertex of the parallelogram so obtained is a point on the line representing the sum of the two numbers. In the figure we have represented the addition 3 + 1 = 4.

18



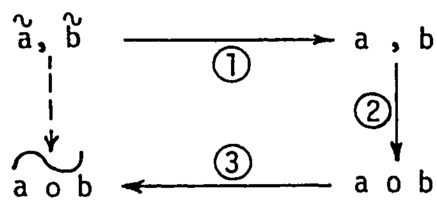
Note that if we were to take any two other representatives (points) of the same number, the result would be the same - i.e. we would obtain another representative (point) of the same number.

A Fourth Methodological Note

IV. Let S be a set on which an equivalence relation $\sim$ has been defined, and let $\circ$ be a closed binary operation on S. (A closed binary operation is a rule for combining any two elements of S to obtain a unique element of S; e.g. addition on N, but subtraction, although a binary operation is not closed on N.) The equivalence relation $\sim$ defines equivalence classes: thus if $a \in S$ we denote by $\tilde{a}$ the equivalence class to which a belongs, and by $\tilde{S}$ the set of equivalence classes. It now seems reasonable that corresponding to $\circ$ on S, we should define $\tilde{\circ}$ on $\tilde{S}$ by

$$\tilde{a} \mathbin{\tilde{\circ}} \tilde{b} = \widetilde{a \circ b} .$$

(It is usual to denote $\tilde{o}$ also by o , once one has got used to the idea.)
That is to combine two equivalence classes, we take a representative from each, combine the representatives, and identify the class to which the result belongs.
The procedure is described by the following diagram



The only problem is in the first step, since it is not unique. If $a_1 \varepsilon \tilde{a}$ and $b_1 \varepsilon \tilde{b}$, we would just as well have obtained

$$\tilde{a} \, \tilde{o} \, \tilde{b} = \widetilde{a_1 \, o \, b_1}$$

and there is no a priori reason to suppose that

$$\widetilde{a_1 \, o \, b_1} = \widetilde{a \, o \, b} \, .$$

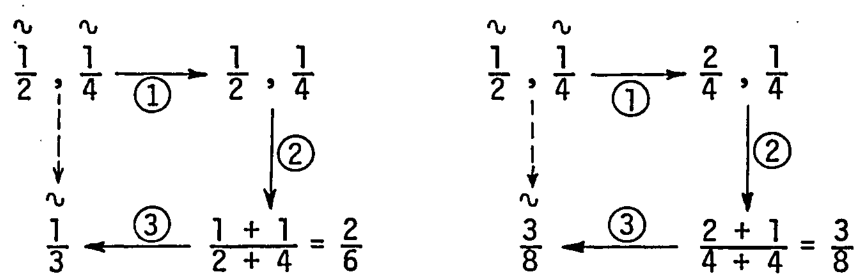
It is precisely this that we checked at the end of the first part of section 5 above. The point can be made clear by returning to our example of fractions and rational numbers.

In the set of fractions we can define all sorts of binary operations. For example, consider the following two

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \qquad\qquad \frac{a}{b} \, o \, \frac{c}{d} = \frac{a + c}{b + d} \, .$$

It is a technical exercise to prove that the first, the usual addition of fractions, can be transferred to the addition of rational numbers, since any equivalent fractions will produce an equivalent result. The second,
18 however, which is a common student error for addition,

cannot be transferred to give the combination of rational numbers, as the following numerical examples show.



and $\tilde{\frac{1}{3}} \neq \tilde{\frac{3}{8}}$, so the combination of $\tilde{\frac{1}{2}}$ and $\tilde{\frac{1}{4}}$ is not uniquely defined.

Thus whenever we define a binary operation between classes of objects by choosing representatives from the classes and combining these, we have to be careful that the definition is independent of the choice of representative.

In our context we wish to combine natural numbers as represented by classes of ordered pairs, by the definition

$$(a, b) + (c, d) = (a + c, b + d) .$$

But there are many pairs equivalent to (a, b) and (c, d), and we have to check that the combination of any equivalent pair always results in a pair equivalent to $(a + c, b + d)$.

Continuation of solutions

6. i) and ii) The natural definition of subtraction would seem to be

$$(a, b) - (c, d) = (a - c, b - d) .$$

But if we choose (2, 5) corresponding to 3 and (8, 9) corresponding to 1, then although 3 - 1 is possible, by our definition we have

$$(2, 5) - (8, 9) = (2 - 8, 5 - 9)$$

which is illegitimate.

So we start again.

(a, b) represents b - a and (c, d) represents d - c, and we can perform the subtraction in N if and only if

$$b - a > d - c ,$$

to obtain $b - a - (d - c) = (b + c) - (a + d)$,

which belongs to N and can be represented by (a + d, b + c).

Hence we define

$$(a, b) - (c, d) = (a + d, b + c) .$$

Now we don't have any subtraction on the right-hand side, so the former trouble has disappeared. But note that the pair (a + d, b + c) represents a natural number if and only if

$$b + c > a + d$$

$$\iff b - a > d - c ,$$

which was our initial condition above - so all is well. It still remains to prove that the definition is valid, i.e. that it is independent of the pairs chosen to represent the natural numbers. This is entirely similar to the proof in section 5.

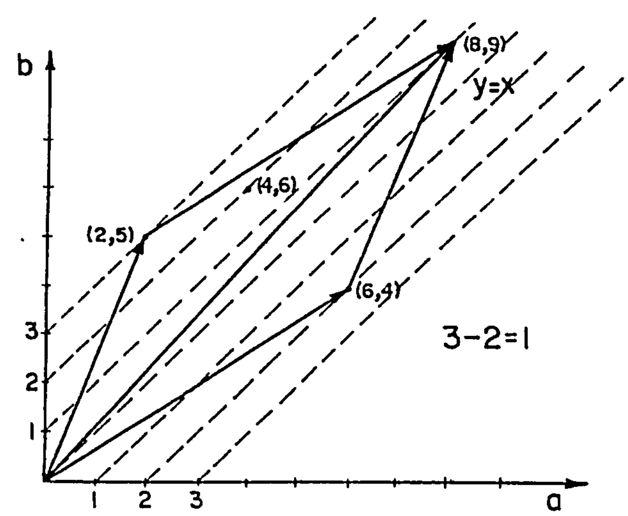
iii) Our definition of subtraction is equivalent to

$$(a, b) - (c, d) = (a, b) + (d, c)$$

(although (d, c) does not represent a natural number, since $d > c$).

Hence we can draw the graphical representation as for addition, except that we shall make use of points "below" the line $y = x$, which do not represent natural numbers. (Note that (d, c) is obtained by reflecting (c, d) in the line $y = x$.)

As we shall see later if (a, b) corresponds to $n \in N$, then (b, a) corresponds to $-n$.



7. In this question we only change representation, but otherwise all remains unchanged. The "new" equation can be written

$$(1, 1 + a) + (1, 1 + x) = (1, 1 + b)$$

or some equivalent form. The solution is

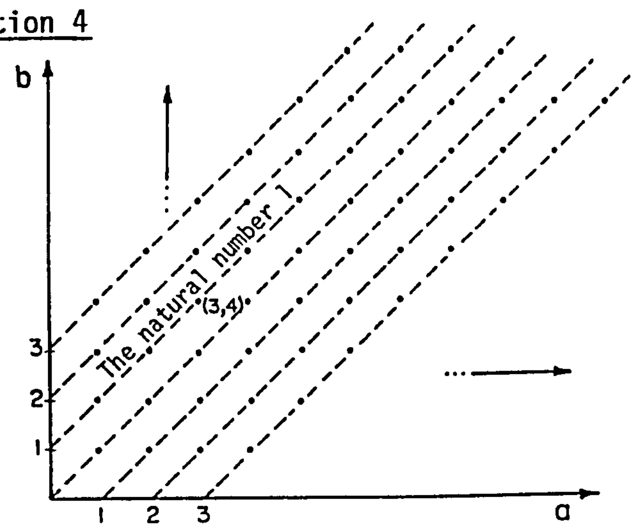
$(1, 1 + x) = (1, 1 + b) - (1, 1 + a) = (2 + a, 2 + b)$ which is equivalent to (a, b).

8. If $a > b$, then (a, b) does not represent a natural number. The definition of equivalence as given in sections 2 and 3 was

$(a, b) \sim (c, d)$ if and only if $a + d = b + c$.

This definition makes sense even if we remove the restriction $b > a$, because we remain within the set N and its arithmetic.
This is the reason we suggested using addition only - because if we had used the alternative form
$(a, b) \sim (c, d)$ if and only if $b - a = d - c$
we would not be able to remove the condition $b > a$.

9. i) <u>Section 4</u>



The graph now extends on both sides of the line $y = x$.

Section 5. Addition can be defined in exactly the same way as before, even though we are now "adding" pairs that do not correspond to natural numbers.

Section 6. Subtraction is also defined the same way, only we remove the condition $b + c > d + a$.

Section 7. We drop the condition $b > a$ and hence obtain the rewritten equation in the form

$$(1, 1 + a) + (x_1, x_2) = (1, 1 + b)$$

Note that we have to write (x_1, x_2) and cannot write $(1, 1 + x)$ as before, because the solution may not now belong to N.

The formal solution of this equation becomes

$$\begin{aligned}(x_1, x_2) &= (1, 1 + b) - (1, 1 + c) \\ &= (2 + a, 2 + b) \text{ by the extension of section 6,}\end{aligned}$$

which is equivalent to (a, b) as before.

ii) Formally we have answered this question in the immediately preceding note. But we have to realise that the solution (a, b) involves a conceptually new entity when $b < a$. See also section 12 ii).

10. The extension in section 8 is very similar to that of Peacock's extension by the principle of permanence of equivalent forms. The difference lies in the fact that we have been very particular to define our extension in terms of the known arithmetic in N, whereas Peacock's principle leads to expressions like $b - a$, which are

essentially meaningless, and are to be manipulated by known rules, also not too carefully specified. As we have seen, this lack of precision can lead us to trouble. Whereas Peacock extends by leaps and bounds our extension is painstaking brick upon brick.

11. All the equivalent ordered pairs of the form (k, n + k), $k \in N$ are denoted by n, as before.
(Intuitively, 0 corresponds to a + x = a, i.e. (a, a).)
So we denote the set of equivalent ordered pairs of the form (k, k), $k \in N$, by 0.
(By definition, (a, b) + (b, a) = (a + b, a + b), which latter we denoted by 0. Hence if (a, b) is denoted by n, (b, a) is denoted by - n.)
So we denote the set of equivalent ordered pairs of the form (n + k, k), $k \in N$ by -n.
Note that because our definitions of addition and subtraction are independent of the ordered pair chosen to form a set of equivalent pairs, we can add and subtract integers.
Note also that in the graphical representation, the line representing n is symmetric, with respect to the line representing 0, to the line representing -n.

12. i) Since the addition and subtraction of ordered pairs always gives another ordered pair, and Z is the set of equivalence classes of all ordered pairs, both $z_1 + z_2$ and $z_1 - z_2$ belong to Z for all $z_1, z_2 \in Z$.

Hence, unlike, N, Z is also closed for subtraction.

ii) The temptation is to write

$$z_1 + x = z_2 \implies x = z_2 - z_1 \in Z \text{ by part i)}$$

But we have only defined addition and subtraction, we have not proved any of their properties - e.g. that addition is commutative and associative. In fact, the proofs of all these properties are simple technical manipulations. For example, we require the property

$$0 + z = z \qquad \text{for all } z \in Z$$

Proof Represent 0 and z by ordered pairs, i.e.

$0 + z$ is represented by $(a, a) + (c, d)$, $a, c, d \in N$

Now, by definition $(a, a) + (c, d) = (a + c, a + d)$ and, by the definition of equivalence, the latter is equivalent to (c, d).

If we write out the solution of $z_1 + x = z_2$ in full, it looks like this

$$z_1 + x = z_2$$

$\implies -z_1 + (z_1 + x) = -z_1 + z_2$

$\implies (-z_1 + z_1) + x = -z_1 + z_2$ associativity of +

$\implies 0 + x = -z_1 + z_2$ $-z_1 + z_1 = 0$

$\implies x = -z_1 + z_2$ $0 + x = x$

$\implies x = z_2 + (-z_1)$ commutativity of +

$x = z_2 - z_1$ addition of $-z_1$ is equivalent to subtraction of z_1.

We content ourselves with the knowledge that each of these properties can be proved, but this should not be taken as asserting that all the properties of arithmetic in N can be transferred to Z. For example, we have already seen that subtraction is not closed in N, but is closed in Z. And when we come to multiplication there will be properties in N which do not extend to Z.

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The Negative Numbers

Worksheet: The formal entry of the negative numbers into mathematics

Part II: Proofs

In this sheet we continue the development of the formal definition of negative numbers. We shall formally prove some of the properties that constituted "stumbling blocks" in the history of negative numbers, precisely because of the absence of mathematical definitions.

Since Part I has been full of technical steps, our first question will review what we have done, summarizing it.

1. Recalling that we use

$\rightarrow$	to denote "corresponds to"
$\sim$	to denote "equivalent to"
$\Rightarrow$	to denote "implies"
$\Leftrightarrow$	to denote "if and only if".

complete the following synopsis:

Ordered pairs $\quad a, b, c, d, \in N$

Sections 1 - 3

$$a + x = b \;,\; x \in N \quad \Rightarrow \quad \underline{\quad > \quad}$$

$$\downarrow$$

$$(\;,\;) \quad \rightarrow \quad \underline{\qquad} \in N$$

$$(a, b) \sim (c, d) \quad \Leftrightarrow \quad \underline{\quad + \quad = \quad + \quad}$$

Addition and subtraction $\quad n, m, a, b \in N$

Sections 5 - 6

$$n + m$$

$$\swarrow \qquad \nwarrow$$

$$(a, a + n) + (b, b + m) \underset{\text{by def}}{=} \underline{(\qquad\qquad , \qquad\qquad)}$$

$$n - m \;,\quad n > m$$

$$\swarrow \qquad \nwarrow$$

$$(a, a + n) - (b, b + m) \underset{\text{by def}}{=} \underline{(\qquad\qquad , \qquad\qquad)}$$

Integers

Section 11

$$a, b \in N \qquad b > a$$

$$n \longrightarrow (a, b)$$

$$0 \longrightarrow \underline{(\;,\;)}$$

$$-n \longrightarrow \underline{(\;,\;)}$$

19

2. Define multiplication in Z , and examine it for validity. (Use the relevant ideas from Part I : ordered pairs, equivalence, etc.)

3. In the Peacock worksheet an extract of an SMSG textbook was to be discussed in the light of Peacock's Principle.

Chapter 7

PROPERTIES OF MULTIPLICATION

7-1. Multiplication of Real Numbers

Now let us decide how we should multiply two real numbers to obtain another real number. All that we can say at present is that we know how to multiply two non-negative numbers.

Of primary importance here, as in the definition of addition, is that we maintain the "structure" of the number system. We know that if a, b, c are any numbers of arithmetic, then

$$ab = ba,$$
$$(ab)c = a(bc),$$
$$a \cdot 1 = a,$$
$$a \cdot 0 = 0,$$
$$a(b + c) = ab + ac.$$

(What names did we give to these properties of multiplication?) Whatever meaning we give to the product of two real numbers, we must be sure that it agrees with the products which we already have for non-negative real numbers and that the above properties of multiplication still hold for all real numbers.

Consider some possible products:

$(2)(3)$, $(3)(0)$, $(0)(0)$, $(-3)(0)$, $(3)(-2)$, $(-2)(-3)$.

(Do these include examples of every case of multiplication of positive and negative numbers and zero?) Notice that the first three products involve only non-negative numbers and are therefore already determined:

$$(2)(3) = 6, \quad (3)(0) = 0, \quad (0)(0) = (0).$$

Now let us try to see what the remaining three products will have to be in order to preserve the basic properties of multiplication listed above. In the first place, if we want the multiplication property of 0 to hold for all real numbers, then we must have $(-3)(0) = 0$. The other two products can be obtained as follows:

$0 = (3)(0)$	
$0 = (3)(2 + (-2))$,	by writing $0 = 2 + (-2)$; (Notice how this introduces a negative number into the discussion.)
$0 = (3)(2) + (3)(-2)$,	if the distributive property is to hold for real numbers;
$0 = 6 + (3)(-2)$,	since $(3)(2) = 6$.

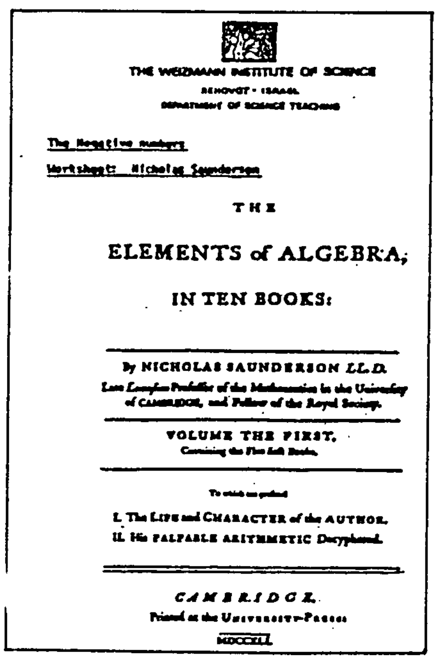
We know from uniqueness of the additive inverse that the only real number which yields 0 when added to 6 is the number -6. Therefore, if the properties of numbers are expected to hold, the only possible value for $(3)(-2)$ which we can accept is -6.

The multiplication property of zero and the distributive law are extended from the positive to the negative numbers in order to conclude the law of signs of multiplication.

In the formal development given here we do not have to make these extensions, we can prove them.

i) Prove that $a \cdot 0 = 0$ for all $a \in Z$

ii) Prove that multiplication is distributive over addition.



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The Negative numbers
Worksheet: Nicholas Saunderson

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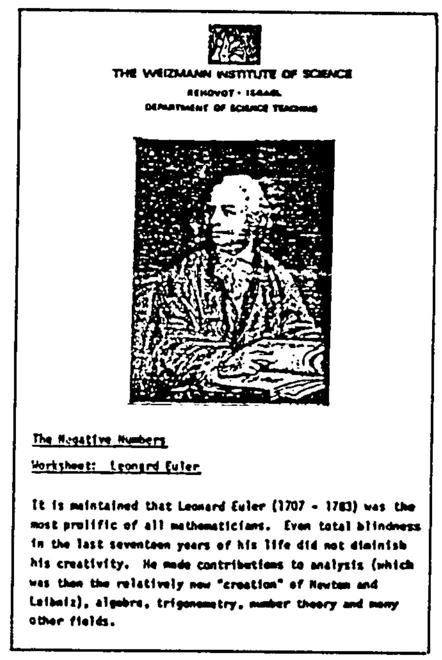

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The Negative Numbers
Worksheet: Leonard Euler

It is maintained that Leonard Euler (1707 - 1783) was the most prolific of all mathematicians. Even total blindness in the last seventeen years of his life did not diminish his creativity. He made contributions to analysis (which was then the relatively new "creation" of Newton and Leibniz), algebra, trigonometry, number theory and many other fields.

4. In the Saunderson worksheet and in the Euler worksheet

19

we saw ways of introducing the multiplication of negative numbers; in particular, the case of two negative factors. The excerpts brought in those sheets were discussed both from the didactical and the mathematical point of view. Didactically we compared the strategies among themselves, and also with the presentations in current textbooks. Mathematically we pointed out some "weaknesses" of the presentations, principally due to the lack of mathematical definition of negative numbers. These weaknesses made it difficult to answer the rejectionists such as Frend and Maseres. Now, at this stage of development of the formal mathematics we are able *to prove*, for instance, that the product of two negative numbers is a positive number.

Prove the "law of signs" of multiplication.

5. In the Euler worksheet we saw how the author may have tacitly used the assumption that the cancellation law can be extended from the positive to the negative numbers.

   Now, again, we can *prove* that this law holds.

   Prove that for $a \neq 0$, $ab = ac \Rightarrow b = c$ for all $b, c \in Z$.

Part III : And so on ...

1. For all $a, b \in Z$ does $ax = b$ have a solution in Z ?

   What do you suggest ? ...

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The Negative Numbers

Answer sheet: The formal entry of the negative numbers into mathematics

Part II: Proofs

1. Ordered pairs $\quad a, b, c, d \in N$

$$a + x = b, \quad x \in N \quad \Rightarrow \quad \underline{b > a}$$

$$\downarrow$$

$$\underline{(a, b)} \quad \rightarrow \quad \underline{b - a \in N}$$

$$(a, b) \sim (c, d) \quad \Leftrightarrow \quad \underline{a + d = b + c}$$

Addition and subtraction $\quad n, m, a, b \in N$

$$n + m$$

$$\swarrow \qquad \nwarrow$$

$$(a, a + n) + (b, b + m) \underset{\text{by def}}{=} \underline{(a + b, a + n + b + m)}$$

$$n - m \qquad n > m$$

$$\swarrow \qquad \nwarrow$$

$$(a, a + n) - (b, b + m) \underset{\text{by def}}{=} \underline{(a + b + m, a + n + b)}$$

Integers $a, b \in N \quad b > a$

$$n \to (a, b)$$

$$0 \to \underline{(a, a)}$$

$$-n \to \underline{(b, a)}$$

2. In order to obtain a definition of multiplication in Z, we go back and look at what it means for those ordered pairs which correspond to natural numbers, where multiplication is already defined.

$$(a, b) \to b - a \in N$$

$$(c, d) \to d - c \in N$$

Hence

$$(a, b)\cdot(c, d) \to (b - a)\cdot(d - c) = bd - ad - bc + ac$$
$$= bd + ac - (ad + bc) \in$$

Thus we *define*

$$(a, b) \cdot (c, d) = (ad + bc, bd + ac)$$

Here again, the validity of this definition depends on two things. The first is that our definition is entirely within the arithmetic of N, which is obviously the case. The second is that it is independent of the pairs we choose to represent the natural numbers:

that is, if we take other pairs, equivalent to those we have here, to show that the result of their multiplication is also an equivalent pair. This is again a technical exercise, which does not make very instructive reading - so we just give the outline of the proof.

Suppose $(a, b) \sim (a_1, b_1)$, i.e. $a + b_1 = a_1 + b$ (1)

and $(c, d) \sim (c_1, d_1)$, i.e. $c + d_1 = c_1 + d$ (2)

we have to show that

$$(a, b)\cdot(c, d) \sim (a_1, b_1)\cdot(c_1, d_1)$$

i.e. $(bc + ad, ac + bd) \sim (b_1c_1 + a_1d_1, a_1c_1 + b_1d_1)$

i.e. $bc + ad + a_1c_1 + b_1d_1 = ac + bd + b_1c_1 + a_1d_1$ (3)

It remains to derive (3) from (1) and (2).

Now, if we drop the constraint $b > a$, $d > c$, we still remain within the arithmetic of N when we define

$$(a, b) \cdot (c, d) = (ad + bc, bd + ac)$$

Thus the definition of multiplication of integers rests solely on addition and multiplication in N.

3. i) By definition, if a is represented by (a_1, a_2) and 0 by by (m, m),

$$(a_1, a_2)\cdot(m, m) = (a_1m + a_2m, a_1m + a_2m)$$

and the latter is just another pair which represents 0.

ii) One has to prove that

$$(a, b)\cdot[(c, d) + (e, f)] = (a, b)\cdot(c, d) + (a, b)\cdot(e, f),$$

which is only a matter of patience.

Note that since we are working with ordered pairs, rather than sets of equivalent ordered pairs, the = sign could be replaced by the $\sim$ sign, which is less restrictive. But this is not necessary here.

4. We saw that pairs of the form $(k, k + n)$ represent the positive number n, for all $k \in N$, and of the form $(k + n, k)$ the negative number $-n$.

Thus if, for example, we wish to prove that "negative x negative = positive", we have to show that

$$(k + n, k)\cdot(\ell + m, \ell)$$

is the form $(p, p + mn)$.

This again is a simple technical exercise

$$(k + n, k)\ (\ell + m, \ell) = k\ell + km + k\ell + n\ell,\ k\ell + km + n\ell + nm + k\ell)$$

$$= (s, s + nm),$$

where $s = k\ell + km + k\ell + n\ell$.

In a similar way the other parts of the rule of signs can be proved.

20

5. $$ab = ac$$

$$\Rightarrow (a_1, a_2)\cdot(b_1, b_2) \sim (a_1, a_2)\cdot(c_1, c_2)$$

(Note that here we have to use $\sim$ and not $=$. Can you explain why?)

$$\Rightarrow (a_2b_1 + a_1b_2, a_1b_1 + a_2b_2)$$

$$\sim (a_2c_1 + a_1c_2, a_1c_1 + a_2c_2)$$

$$\Rightarrow a_2b_1 + a_1b_2 + a_1c_1 + a_2c_2 = a_1b_1 + a_2b_2 + a_2c_1 + a_1c_2$$

(i) $$\Rightarrow a_1(b_2 + c_1) + a_2(b_1 + c_2) = a_1(b_1 + c_2) + a_2(b_2 + c_1)$$

Since $a \neq 0$, either $a_1 > a_2$ or $a_2 > a_1$. Suppose the former, then the last equation above implies

$$(a_1 - a_2)\cdot(b_2 + c_1) = (a_1 - a_2)\cdot(b_1 + c_2) \qquad \text{(ii)}$$

Now $a_1 - a_2 \in N$. Hence we know (from N) that this equation implies

$$b_2 + c_1 = b_1 + c_2 \qquad \text{(iii)}$$

$$(b_1, b_2) \sim (c_1, c_2)$$

$$b = c$$

But if $a_2 > a_1$, then $a_1 - a_2 \notin N$. So the transition from step (ii) to (iii) is to rely on what we wanted to prove. What would you suggest in this case? The answer is to manipulate differently the equation (i).

We have proved some of the arithmetic results in Z only - all the rest, e.g. associativity and commutativity of addition and multiplication, or $a\cdot b = 0 \Rightarrow a = 0$ or $b = 0$, can be proved similarly, or deduced from results already proved. The object was to exemplify how Z and its arithmetic could be constructed out of N, not to give an exhaustive treatment, which after a while becomes boring.

In our view, the value of these questions to us, as teachers is that one should be aware of the mathematical development; not that one should teach in this way.

An approach at the "level" and in the style of Euler or Saunderson would seem to be more appropriate to the classroom. But with a knowledge and appreciation of the formal mathematics, the teacher is aware of the mathematical compromises he is making, and will know which points to emphasize and how to answer students' questions intelligently.

Part III: And so on ...

1. The solution x of $ax = b$ belongs to Z if and only if a divides b. (Note that division in Z has not been defined here - but can be - by methods similar to the above.)

   When a does not divide b, we are in exactly the same position as we were in N with the equation $a + x = b$, when $a > b$. Hence we can play the same game again. Represent x by (a, b). When does (a, b) represent $x \in Z$? what is the definition of equivalent pairs now? Etc. Following exactly similar procedures we can construct Q, the set of rationals.

We still will not have all the positive numbers, because the irrationals are still missing. This is a different story - to be told in another set of worksheets.

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The Negative Numbers

Worksheet: Summary

Part I

The following extract is taken from *Mathematical Thought from Ancient to Modern Times* (1972), a large and comprehensive work by Morris Kline.

In the extract several points in the history of negative numbers are mentioned, including names of mathematicians, their works and some of their arguments.

L. Euler

A. De Morgan

By 1700 all of the familiar members of the system—whole numbers, fractions, irrationals, and negative and complex numbers—were known. However, opposition to the newer types of numbers was expressed throughout the century. Typical are the objections of the English mathematician Baron Francis Masères (1731–1824), a Fellow of Clare College in Cambridge and a member of the Royal Society. Masères, who did write acceptable papers in mathematics and a substantial treatise on the theory of life insurance, published in 1759 his *Dissertation on the Use of the Negative Sign in Algebra.* He shows how to avoid negative numbers (except to indicate the subtraction of a larger quantity from a smaller one), and especially negative roots, by carefully segregating the types of quadratic equations so that those with negative roots are considered separately; and, of course, the negative roots are to be rejected. He does the same with cubics. Then he says of negative roots,

> ...they serve only, as far as I am able to judge, to puzzle the whole doctrine of equations, and to render obscure and mysterious things that are in their own nature exceeding plain and simple.... It were to be wished therefore that negative roots had never been admitted into algebra or were again discarded from it: for if this were done, there is good reason to imagine, the objections which many learned and ingenious men now make to algebraic computations, as being obscure and perplexed with almost unintelligible notions, would be thereby removed; it being certain that Algebra, or universal arithmetic, is, in its own nature, a science no less simple, clear, and capable of demonstration, than geometry.

Certainly negative numbers were not really well understood until modern times. Euler, in the latter half of the eighteenth century, still believed that negative numbers were greater than ∞. He also argued that $(-1)\cdot(-1) = +1$ because the product must be $+1$ or -1 and since $1\cdot(-1) = -1$, then $(-1)\cdot(-1) = +1$. Carnot, the noted French geometer, thought the use of negative numbers led to erroneous conclusions. As late as 1831 Augustus De Morgan (1806–71), professor of mathematics at University College, London, and a famous mathematical logician and contributor to algebra, in his *On the Study and Difficulties of Mathematics*, said, "The imaginary expression $\sqrt{-a}$ and the negative expression $-b$ have this resemblance, that either of them occurring as the solution of a problem indicates some inconsistency or absurdity. As far as real meaning is concerned, both are equally imaginary, since $0 - a$ is as inconceivable as $\sqrt{-a}$."

De Morgan illustrated this by means of a problem. A father is 56; his son is 29. When will the father be twice as old as the son? He solves $56 + x = 2(29 + x)$ and obtains $x = -2$. Thus the result, he says, is absurd. But, he continues, if we change x to $-x$ and solve $56 - x = 2(29-x)$, we get $x = 2$. He concludes that we phrased the original problem wrongly and thus were led to the unacceptable negative answer. De Morgan insisted that it was absurd to consider numbers less than zero.

Questions

1. Complete the information missing in the following summary table.

We shall use the following "notation":

------ to indicate that the information can be found in the extract from Kline.

? to indicate that the extract does not include the information, but it can be found in previous worksheets in this series.

(?) to indicate that the extract does not include the information and it is also not to be found in the worksheets.

2. Compare Euler's first referenced argument (i) with the Euler extract brought in the answer sheet: *Contradictions in the use of negative numbers*.

3. Compare Kline's quotation of De Morgan with the two quotations brought in the answer sheet: *The last of the opposition - Frend* (p. 5-6 and 16-17).

4. What can you conclude from your answers to questions 2 and 3?

MATHEMATICIAN	TITLE OF HIS BOOK, PAPER, ETC.	HIS ARGUMENT CONCERNING NEGATIVE NUMBERS
F. Maseres (____-____)	(i) ____________________ ____________________ (ii) ?	(i) "... to avoid negative numbers ... segregating the types of quadratic equations so that ____ ____ ____." (ii) "... when writers of Algebra talk of the negative roots of an equation, they, in fact, jumble two different equations together, ...__ ____?______?______?____ ____?______?______?____."
L. Euler (? - ?)	(i) ? (ii) ?	(i) "Euler, in the latter half of the eighteenth century, still believed that ______________" (ii) "... (-1)·(-1) = +1 because _______________"
A. De Morgan (____-____)	____________________	"... De Morgan ..., said:..'either of them [a negative or an imaginary] occurring as the solution of a problem indicates_________________________________, ... De Morgan illustrates this by means of a problem ____"

Part II

In this sequence of worksheets we came across the names of quite a number of people, among them famous mathematicians. We discussed their approach and their position on negative numbers.

In the following we bring,

I. Some of these names:

Viète, Descartes, Arnauld, Wallis, Saunderson, Euler, Frend, Maseres, Peacock, De Morgan, Hamilton.

II. A suggested division of the development of the negative number concept into stages:

- *non recognition of negatives,*
- *recognition of negatives as roots of equations,*
- *use of negatives with reservations because of the "contradictions" arising from their use,*
- *the free use of negatives and their entry into textbooks without mathematical definitions,*
- *opposition to negatives,*
- *attempt to give a mathematical foundation of the negatives,*
- *formal mathematical treatment.*

III. A rough chronological table:

<u>Questions</u>

5. i) Identify the various stages in the chronological table.

   ii) Insert the name of each mathematician in the appropriate stage.

   III) Can you suggest a different division of the development of the negative number concept into stages?

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1983

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The Negative Numbers

Answer sheet: Summary

Part I

The extract from *Mathematical Thought from Ancient to Modern Times* (1972) by Kline brings various "facts" from the history of the negative numbers, which we met in previous worksheets and answer sheets.

1. In the following we bring the complete table. Note that, in the cases in which we used the symbol ? to indicate that the information can be found in previous worksheets, we bring here the reference. The symbol Ⓚ is used to denote that the quotation is from the Kline extract.

MATHEMATICIAN	TITLE OF HIS BOOK, PAPER, ETC.	HIS ARGUMENT CONCERNING NEGATIVE NUMBERS
F. Maseres (1731-1824)	(i) *Dissertation on the Use of the Negative Sign in Algebra* (1759) (ii) Appendix to *The Principles of Algebra* (1796) by W. Frend. (See full quotation in the answer sheet: *The Last of the Opposition - W. Frend*)	(i) He shows how to avoid negative numbers (except to indicate the subtraction of a larger quantity from a smaller one), and especially negative roots, by carefully segregating the types of quadratic equations so that those with negative roots are considered separately; and, of course, the negative roots are to be rejected. (K) (ii) when writers of Algebra talk of the negative roots of an equation, they, in fact, jumble two different equations together. . . Thus, for example, they would say, that the quadratick equation $xx + 4x = 320$, has two roots, to wit, the positive, or affirmative, root, $+ 16$, and the negative root, $- 20$. But this latter number, 20, is, in truth, the root of a different equation, to wit, of the equation $xx - 4x = 320$.
L. Euler (1707-1783)	(i) (?) (ii) *Elements of Algebra* (1770) (The 1898 edition is quoted in the answer sheet *Euler*)	(i)-(ii) Euler, in the latter half of the eighteenth century, still believed that negative numbers were greater than ∞. He also argued that $(-1)\cdot(-1) = +1$ because the product must be $+1$ or -1 and since $1\cdot(-1) = -1$, then $(-1)\cdot(-1) = +1$. (K)
A. De Morgan (1806-1871)	*On the Studies and Difficulties of Mathematics* (1831- the 1898 edition is quoted in the answer sheet: *The Last of the Opposition - W. Frend*)	The imaginary expression $\sqrt{-a}$ and the negative expression $-b$ have this resemblance, that either of them occurring as the solution of a problem indicates some inconsistency or absurdity. De Morgan illustrated this by means of a problem. A father is 56; his son is 29. When will the father be twice as old as the son? He solves $56 + x = 2(29 + x)$ and obtains $x = -2$. Thus the result, he says, is absurd. But, he continues, if we change x to $-x$ and solve $56 - x = 2(29 - x)$, we get $x = 2$. (K)

2. In the Euler first referenced argument we read that "Euler, in the latter half of the eighteenth century, still believed that negative numbers were greater than ∞." In the answer sheet *Contradictions in the use of negative numbers* we read,in a paragraph from Euler's *Differenzial Rechnung* (1790), the opposite: he destroys the argument of those who concluded that the negatives are greater than infinity.

3. Kline's conclusion of De Morgan's quotations is: "De Morgan insisted that it was absurd to consider numbers less than zero". In the answer sheet *The Last of the Opposition - W. Frend* we read two paragraphs by De Morgan. The first, taken from *A Budget of Paradoxes* is an answer to Maseres' rejection of negatives, in which he says "...the great difficulty of the opponents of algebra lay in want of power or will to see extension of terms...". The second is the same paragraph as that quoted by Kline (with the father-son problem) but with a commentary by De Morgan not brought by Kline, which is: "In either case, however, the result must be interpreted, and a rational answer to the question may be given."

4. The comparisons made in the previous sections lead us to point out how difficult it is to study and/or write history of mathematics, if one wants to be "loyal" to the truth, and to take into account "all the facts". Especially in a monumental work such as that of Kline, it is virtually impossible for the author to check every statement back to its source.

Nevertheless, some advice may be given in order to optimise reliable information, when doing history of mathematics.

We quote from Spalt*(1983):

> Bei der Auseinandersetzung mit mathematischen Texten (und nicht nur solchen!) gilt es, *die goldenen Regeln der Geschichtsschreibung* zu beachten. Deren erste lautet: *Geh' zu den Quellen!* Hier also: Lies die Originaltexte!
> ...
> Nun müssen wir uns der *zweiten goldenen Regel der Geschichtsschreibung* erinnern: *Beachte die Zeitgebundenheit!* Hier also: Bleibe im Textzusammenhang!

Whenever we break one of these rules, which inevitably occurs, we should accept the information with gentle reserve.

We would suggest a further rule, which we hope we have not ourselves broken too often in this series of worksheets. Even after studying the primary source, and making sure that we have understood the context, historical reading or writing involves interpretation, conscious or unconscious. Thus one should be careful not to overstress one's own opinion - that is, to leave a certain measure of self doubt. Historical facts are often not as black and white as they seem. Consider the following.

Kline writes:

> Masères, who did write acceptable papers in mathematics and a substantial treatise on the theory of life insurance,...

Which leaves us with a favourable impression of Maseres work on life insurance.

Now compare De Morgan** on the same work.

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* Spalt D.D., Eine längst fällige, wenngleich unnötige Rehabilitation Cauchys, *Der Mathematik-Unterricht*, 29,4, 1983, p. 60-76.

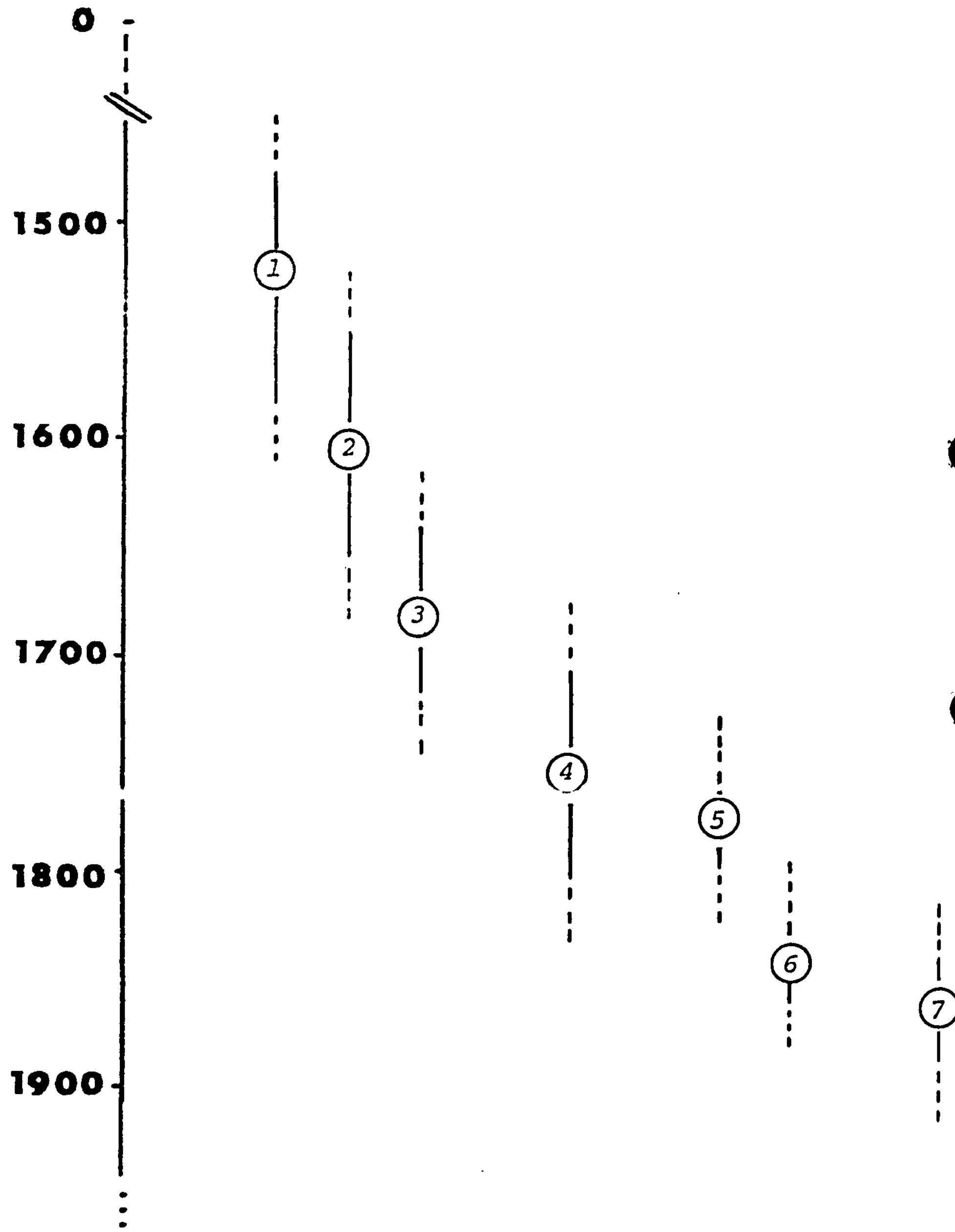
** De Morgan A., *A Budget of Paradoxes*, Vol I. The Open Court Pub., 1915, p. 206

The *Doctrine of Life Annuities*[9] (4to, 726 pages, 1783) is a strange paradox. Its size, the heavy dissertations on the national debt, and the depth of algebra supposed known, put it out of the question as an elementary work, and it is unfitted for the higher student by its elaborate attempt at elementary character, shown in its rejection of forms derived from chances in favor of *the average,* and its exhibition of the separate values of the years of an annuity, as arithmetical illustrations. It is a climax of unsaleability, unreadability, and inutility. For intrinsic nullity of interest, and dilution of little matter with much ink, I can compare this book to nothing but that of Claude de St. Martin, elsewhere mentioned, or the lectures *On the Nature and Properties of Logarithms,* by James Little,[10] Dublin, 1830, 8vo. (254 heavy pages of many words and few symbols), a wonderful weight of weariness.

The stock of this work on annuities, very little diminished, was given by the author to William Frend, who paid warehouse room for it until about 1835, when he consulted me as to its disposal. As no publisher could be found who would take it as a gift, for any purpose of sale, it was consigned, all but a few copies, to a buyer of waste paper.

Part II

5. i-ii)



1. *Non recognition of negatives: Viete*
2. *Recognition of negatives as roots of equations: Descartes*
3. *Use of negatives with reservations because of the "contradictions" arising from their use: Arnauld, Wallis*
4. *Free use of negatives and their entry into "textbooks" without mathematical definitions: Saunderson, Euler*
5. *Opposition to negatives: Frend, Maseres*
6. *Attempt to give a mathematical foundation to the negatives: Peacock, De Morgan, Hamilton(?)*
7. *Formal mathematical definition: Hamilton(?)*

iii) Note that the above is only a suggested division into stages based on the worksheets in this series. There may be others, especially if we take into account the early use of negatives, for instance, by the Hindus or by Fibonacci. Also other names can be added. Finally, other divisions and/or subdivisions could be made.

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The Irrational Numbers

A source-work collection
for in-service and pre-service
teacher courses

A. Arcavi
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October 1984

THE WEIZMANN INSTITUTE OF SCIENCE
REHOVOT · ISRAEL
DEPARTMENT OF SCIENCE TEACHING

The Irrational Numbers

Introduction and commentaries to the set of worksheets on some episodes in the historical development of the irrational numbers

The attached materials are designed for use in workshops for pre and in-service teacher training, in particular for teachers in junior high schools. Although the sheets have a single topic, viz. the irrational numbers, many general points concerning mathematical activity and its teaching are included (see the individual commentaries).

Thus the general aims of this and other sequences of worksheets is:

1- to *improve* the teachers' *mathematical knowledge* of topics included in the curriculum, and that in a way which motivates the teacher to reconsider topics previously studied but possibly imperfectly understood;

2- to *enrich* the teachers' *mathematical background* to topics in the curriculum;

3- to allow opportunity for the *discussion of relevant didactics* and to consider them in relation to the pure mathematics of the topics concerned;

4- to create a *reasonable image of mathematics* and mathematical activity as a human endeavour. In particular, to create an awareness of the history of topics included in the curriculum.

In general, the worksheets in this sequence have the following form:

- a brief biographical-chronological introduction in order to set the historical scene,
- an historical source; as far as possible a primary source,
- leading questions on the source material and on mathematical and didactical consequences thereof.

To each worksheet an extensive discussion of the solutions, and points arising from them, is given in the respective *answer sheet*. Thus, the answer sheets contain not only the detailed solutions to the questions, but further source material, background, historical and mathematical information. Both the worksheets and the answer sheets are designed as learning materials.

At the present time, this sequence contains 6 worksheets which are intended to be worked in the following order:

1- The Pythagoreans.
2- Euclid and the *Elements*.
3- Irrationals in the 16th and 17th centuries.
4- Rafael Bombelli.
5- Nicholas Saunderson.
6- Dedekind and the definition of the irrationals.

We have used the sheets in the following:

i- In-service workshops.
ii- Pre-service courses.
iii- Correspondence course.
iv- Single worksheets in other workshops.

The work is usually guided by a tutor. Participants are given the first worksheet. They work in groups or individually, with the tutor's "interference" if necessary. Then a collective guided discussion takes place (except for iii-), and the answer sheet is distributed. And so on.

For use with Israeli teachers, we translated the sources in the worksheets freely into Hebrew and gave both the original with the translation, in order to encourage the student to read both (if he understood the original language, if not he could look at it and get some flavour of the period by the form of the print, its elaboration, etc., or even try to identify key words etc. from the translation.) Also the original should be available in cases of misunderstanding attributable to mistranslation. In the present English version there are some extracts in languages other than English, and we include either an "authorized" English translation (for example, Bombelli) or only the English version (for example, Dedekind). Also there are some extracts for which we did not find an "authorized" English translation, in which case we have brought the original only. For English speaking users of these materials, it may be advisable to translate these latter texts freely into English, alongside the original.

This series is not a text. Therefore there is a need for the tutor using the sheets, to add comments, to be in a position to answer questions, and to have in mind the overall structure of the set.

The stages in the history of irrational numbers, represented in these worksheets, are far from being complete. There are certainly, further sheets which could be added, but those here presented give, in our view, the major highlights in the general development of the irrationals. In the following we shall comment briefly on the content of each worksheet, how to link them and some of the points we had in mind when writing and using the worksheets.

Commentary to the individual worksheets

1- The Pythagoreans

This worksheet presents a little from the world of the Pythagoreans: figurate numbers, the pentagram and commensurable line-segments. Then the crisis in the Pythagorean doctrine caused by their realisation of the existence of incommensurable line-segments, is discussed (with two different approaches). Only secondary sources are used in this sheet. This sheet can be enriched by "ramifications"; for example, discussion of several ways to prove the irrationality of $\sqrt{2}$, more details on the golden section, etc.

2- Euclid and *The Elements*

A further stage in the history of the irrationals is the "legitimization of incommensurability". In this context, Euclid's famous masterpiece is presented briefly, and an interesting definition of irrationality is discussed.

3- Irrationals in the 16th and 17th centuries

The (almost) 2000 year jump, can be justified by the sparsity of historical sources that can offer something new, relevant to the development of the irrationals. Nevertheless, it should be worthwhile to refer somehow to this long period. In our workshops, for example, we brought an Hebrew source by Maimonides (1135-1204), one of the greatest Rabbis of all times, codifier, philosopher, physician and astronomer. In his commentary on the Mischna (Eruvin), he discusses the nature of the ratio between the circumference of a circle and its diameter. He states:

יש לך לדעת כי יחס אלכסון העגולה אל המסבב אותה בלי ידוע וא"א לדברו לעולם באמת וחסרון
זו ההשגה אינה מאתנו כמחשבת הכת הנקראת גהלי"ה אבל הוא בטבעי זה הדבר בלי ידוע ואין
במציאותו שיושג אבל (ידוע)[יודע] זה בקרוב וכבר חברו חכמי התשבורת לזה מאורים לידע יחס
האלכסון אל המסבב בקרוב ודרך המופת בזה הקרוב אשר עליו סומכין חכמי החכמות הלמודיות
הוא יחס האחד לשלשה ושביעית וכל עגולה שיהיה באלכסון שלה אמה יהיה בהיקפה ג' אמות
ושביעית בקרוב ולפי שזה לא יושג לעולם אלא בקרוב לקחו הם בחשבון הגדול ואמרו כל שיש
בהיקפו ג"ס יש בו רחב טפח וסמכו ע"ז במה שהוצרכו אליו מן המדידה בתורה ·

which (very briefly) means that "its essence is unknown and cannot be conceived... only by approximation". And then he gives $3\frac{1}{7}$ as an approximate value.

This concern about the nature of irrational numbers became more widespread after decimal fractions were developed in the 16th century, by Simon Stevin (1548-1620), and others, and this is the topic of the present worksheet. Arguments in favour and against irrationals are brought, and it is interesting to note that some of those problems, still remain to trouble students today.

4-5 Rafael Bombelli - N. Saunderson

Once the conflict about irrationals had abated (but not disappeared, in the absence of a mathematical definition), we can consider the next stage as the stage of rational approximations to irrationals. (It should be noted that there were mathematicians who had treated this topic before this period also.)

Two methods of approximation are discussed and compared: Bombelli's and Saunderson's. The first is not so well known by teachers, the second is the most well known (from the algorithmical point of view, but not its justification).

6- Dedekind and the definition of irrationals

The following are examples of stages (between the fifth and sixth worksheets) not treated in this series:

- The rationality or irrationality of some given numbers (for example, the interesting case of π - Lambert's proof of its irrationality in 1766.)
- The existence of algebraic and transcendental irrationals (and also special cases, such as Lindemann's proof of the transcendence of π in 1882, or the existence of number whose transcendence is not yet proved.)

We suggest that these points should at least be mentioned verbally in connection with the Dedekind worksheet. In this worksheet, we bring the original development of the definition, which seems to be easier than its elaboration appearing in many later texts.

Some authors* note the similarity between the Eudoxian theory of proportion (in Euclid's *Elements*, Book V) and Dedekind's definition of a real number as a cut. This may be interpreted as a suggestion that Dedekind got his inspiration from Eudoxus theory, either consciously or not.

In this worksheet, the four elementary operations on irrationals are also defined, and statements like $\sqrt{3} \cdot \sqrt{2} = \sqrt{6}$ are proved.

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* See for example:
Flegg G., *The real numbers,* (History of Mathematics AM 289, Unit 2) The Open University Press, 1974, p.23.

Final Comment

We do not see these sheets as final or definitive. As we find new and or "better" sources, or in the light of experience in the use of the sheets, we make alterations, corrections, replacements, additions, etc.

Other sequences are also in process of development, both with the intention of improving the development per se, and evaluating their implementation.

We would welcome comments from anyone reading these worksheets.

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October 1984.

THE WEIZMANN INSTITUTE OF SCIENCE
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The Irrational Numbers

Worksheet: The Pythagoreans

One of the most famous names in the history of ancient Greek mathematics is that of Pythagoras, who lived in the sixth century B.C.E. The Pythagorean brotherhood was founded in southern Italy, and engaged in agriculture and scientific and religious philosophy.

It is difficult to asses accurately the contribution they made to mathematics, since there is hardly any contemporary written evidence, and many of the things attributed to the Pythagoreans, are shrouded in legend and hearsay.

Apparently, the basis of their philosophy was *number*. Matter was susceptible to change and therefore not fundamental - fundamental was the numerical aspects in which matter was revealed. They believed that the ordered universe existed because of its harmony, which was expressed by simple numerical relations.

Content of this worksheet

First we learn a little about the world of the Pythagoreans: figurate numbers, properties of the pentagon, commensurable line-segments. Then we discuss the crisis in the

Pythagorean doctrine, when they came up against a discovery which contradicted their world view.

As mentioned above, we do not have source material from that time, therefore we shall use secondary sources; that is, material from the histories of mathematics. In the process, we shall bring two different theories of how the Pythagoreans came upon the discovery which rocked their world.

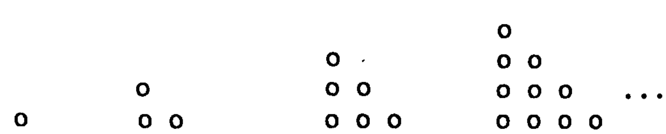
<u>From the Pythagorean world</u>

<u>Figurate numbers</u>

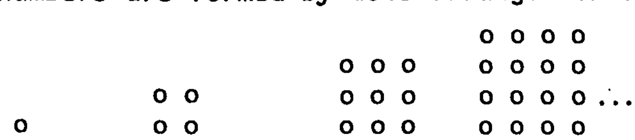
In ancient Greece, the decimal number system, with which we are familiar, did not exist. There is no doubt that this is one of the reasons for the fact that the development of arithmetic and algebra lagged far behind that of geometry. A number was denoted by a letter of the alphabet. In addition, apparently, the Pythagoreans represented numbers by dots. This form of representation is the source of the concept of *figurate numbers*.

The following are some examples of figurate numbers.

- Triangular numbers are represented as follows.



- Square numbers are formed by dots arranged in a square.



- Rectangular numbers take the form of a rectangle whose length exceeds its breadth by one dot.

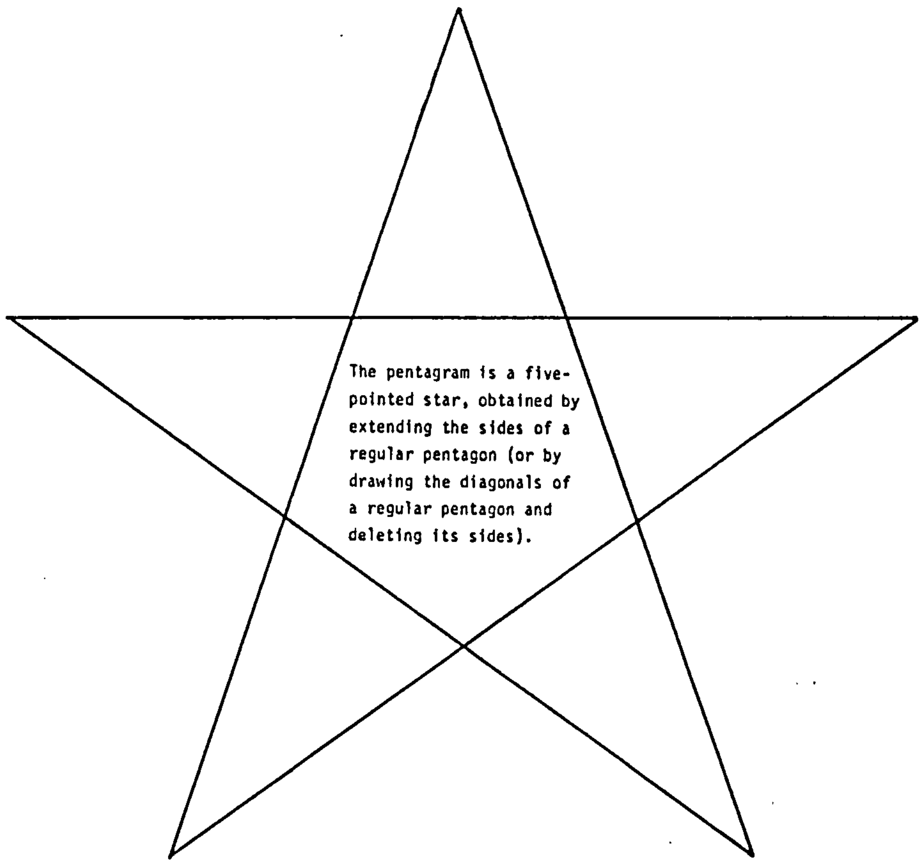
```
                              o o o o o
                    o o o o   o o o o o
          o o o     o o o o   o o o o o   ...
o o       o o o     o o o o   o o o o o
```

Questions

1. Show by drawing
   a) A square number (n^2) is equal to the sum of all the odd numbers which precede n.
   b) The sum of two consecutive triangular numbers is a square number.
   c) The relation between a rectangular and a triangular number.
2. What the Pythagoreans showed using dots, we can prove using the algebraic techniques available to us. Prove the results in the previous question.

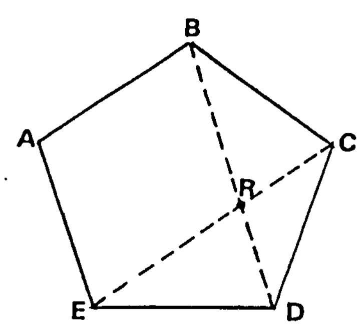
The pentagon

The Pythagoreans, by means of numerical ratios, found particular expression for their concept of harmony in geometry. Apparently, they knew the properties of various regular polygons, including the pentagon and the pentagram.



The Pentagram - Badge of the Pythagoreans

3. ABCDE is a regular pentagon.
   a) Show that ΔRCD is isosceles.
   b) Show that the angle CRD is equal to the angle A (or any other angle of the pentagon).
   c) Show that BC = BR.



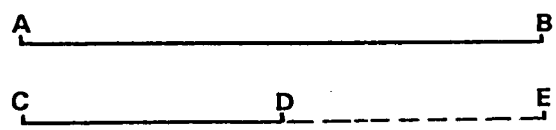
We shall have cause to refer to these results again later.

Common Measure

The Pythagoreans knew how to find the greatest common measure, by the method of mutual subtraction.

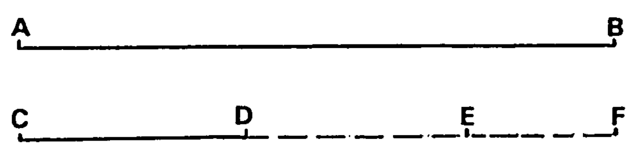
For example,

a)

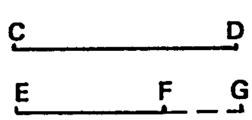


If CD = DE, AB = 2CD. In this case CD is the greatest common measure, and the ratio of CD to AB is the same as the ratio of 1 to 2.

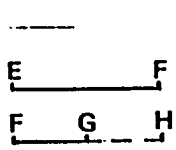
b)



In this case CD is not the common measure. So the process continues



Again EF is not a common measure, so we continue



This time we have found the greatest common measure, and we denote FG by d. Hence

$$CD = 3d$$

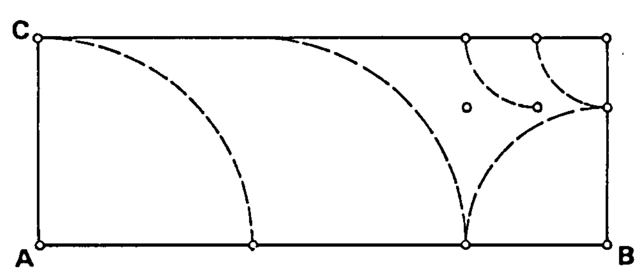
$$AB = 8d$$

and the relation between CD and AB is the same as the relation of 3 to 8.

Note: We have exemplified the process of finding the greatest common measure by drawing. Drawing is, of course, not accurate and will have to be replaced by geometrical reasoning.

Question

4. If we draw the two segments as sides of a rectangle, there is another interesting way of illustrating the process of finding the greatest common measure*. (The dashed curves are drawn with a compass.)



Denote the greatest common measure by a, and mark it appropriately on the figure. Then fill in the following blanks.

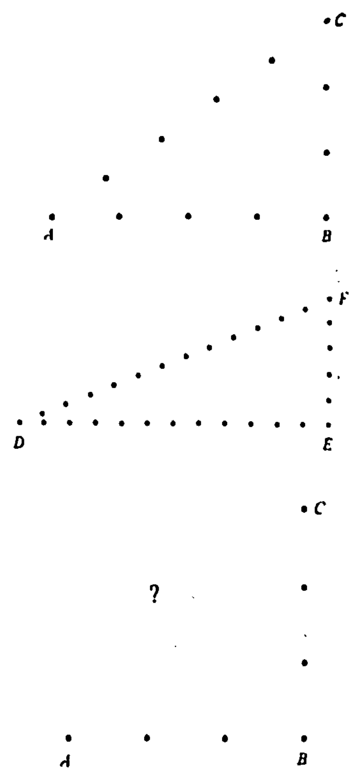
AB = ____ a ,   AC = ____ a

* See for example,
Artmann B. & Seeger V., Geschichte, Geometrie und Irrationalzahlen. Drei Stunden in der Klasse 9. *Der Mathematikunterricht*, 1982, 28, (4), p. 20-9.

The discovery of incommensurable segments

Can we always find a common measure for any two segments? Apparently the Pythagoreans at one time thought that one could.

The following extract* describes the crisis caused by the discovery of incommensurable segments. It contradicted their view of harmony in the universe, and thus began the history of the irrational numbers.



We have seen that the Pythagoreans believed that everything, whether in geometry or in human affairs, could be explained in terms of number. By this they meant whole numbers represented by dots or finite indivisible elements of some sort. Since $3^2 + 4^2 = 5^2$; $5^2 + 12^2 = 13^2$; etc., we can (from the Pythagorean theorem) represent the sides and diagonals of certain rectangles by means of equally spaced dots. We can also in such cases express the *ratio* between diagonal and side as the ratio of a pair of whole numbers. Thus, $AC/AB = 5/4$, $DF/DE = 13/12, \ldots$. In such cases we say that the side and diagonal are *commensurable*.

Exactly at what stage the Pythagoreans became aware of the existence of pairs of magnitudes which could not be compared by means of the ratios of whole numbers in this way we do not know. There is a story that a certain Hippasus of Metapontum (*c.* 400 BC) was expelled from the brotherhood and subsequently drowned at sea as a judgement on him for discovering or disclosing the existence of such quantities. For example, in any square, if AC is the diagonal and AB is the side, can we write $AC/AB = m/n$, where m and n are whole numbers? Aristotle refers to a proof that the diagonal and side of a square are *incommensurable* (i.e. $AC/AB \neq m/n$, where m and n are whole numbers) by means of *odds* and *evens*, and since the properties of odd and even numbers played a central role in Pythagorean thinking it is highly probable this was the proof they found: Let $AC/AB = m/n$, where m and n are whole numbers with no common factor; it follows that m and n cannot *both* be *even* numbers. Since $AB = BC$ and $AC^2 = AB^2 + BC^2 = 2AB^2$, then $AC^2/AB^2 = 2 = m^2/n^2$, $2n^2 = m^2$, and m^2 is *even*; but, if m^2 is *even*, then m is even (since the square of an *odd* number is *odd*). Let $m = 2r$, $m^2 = 4r^2 = 2n^2$, $2r^2 = n^2$, n^2 is *even* and n is also *even*. •••

The proof we have given is one of the easiest example of *proof by contradiction* or *reductio ad absurdum*.[1] Such proofs were constantly used in Greek mathematics and indeed some of the finest proofs in mathematics today take this form. •••

According to Plato, Theodorus of Cyrene (*c.* 390 BC), proved that the sides of squares of areas $3, 5, \ldots, 17$ are incommensurable with the sides of squares of area 1. Explanations as to why Theodorus stopped at 17 vary. •••

It follows that a number system consisting of whole numbers and ratios of whole numbers was seen to be insufficient to represent relations between continuous quantities such as line segments, surfaces and volumes. ••• Very briefly, the impact of all this was that formal Greek mathematics turned away from the arithmetic emphasis of the Pythagoreans and became entirely *geometric* •••

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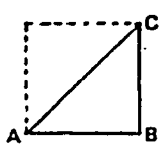
* Taken from

Baron, M.E., *Greek Mathematics*. (AM 289 C1) The Open University Press, 1974, p. 23-24.

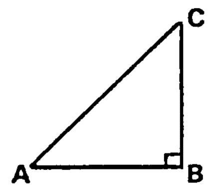
Questions

5.In the following table we bring the "Greek proof" that the diagonal of a square is incommensurable with its side - or, in other words, that $\sqrt{2}$ is irrational. Complete the argument in the table as indicated.

Data	Required to prove	Proof	Reason
The square  	The diagonal and side of the square are segments without ________ ______________	Either $\frac{AC}{AB} = \frac{m}{n}$ or $\frac{AC}{AB} \neq \frac{m}{n}$ Suppose $\frac{AC}{AB} = \frac{m}{n}$	
AB = BC $AB^2 + BC^2 =$ ---- That is $2\ AB^2 =$ _____	That is, integers m and n do not exist, such that $\frac{AC}{AB} = \frac{m}{n}$	m and n are integers with no common factor. $\frac{AC^2}{AB^2} = 2$ $\Downarrow$ | | |	→ ____

6. In the extract it is stated that Theodorus of Cyrene proved that the sides of squares of areas 3, 5, ..., 17 are incommensurable with the sides of the square of area 1. We can construct the corresponding segments using a compass and an initial triangle.

i) Draw an isosceles right-angled triangle - as in the figure.



We take AB as the unit of measurement; i.e., we denote its length by 1. What is the length of AC?

ii) On AC, construct the perpendicular CC_1 of length 1. Join AC_1. What is the length of AC_1?

iii) Continue the same construction of unit perpendiculars C_1C_2, C_2C_3, etc. to obtain a sort of spiral, until the drawing begins to overlap. What is the length of each of AC_2, AC_3, ...?

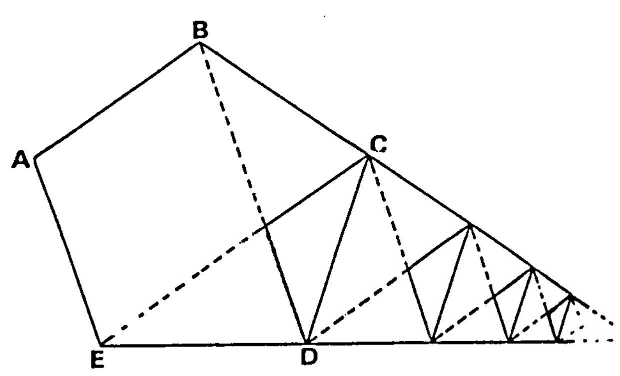
iv) There are historians who maintain that Theodorus knew the proof for a square of area 2 (i.e. that $\frac{m}{n} \neq \sqrt{2}$).

Assuming that he used proofs similar to that for 2, reconstruct (in modern symbolism) the proofs for 3, 5 and 8.

There is a theory* that the relation of the side to the diagonal of a square was not the first instance of incommensurability, which the Greeks met. According to this view, the problem first occurred in the case of a pentagon. In the pentagon, it is geometrically simpler to show that there are two incommensurable segments. (Remember that the above proof for the square looks simple because of our use of modern notation, and the argument is essentially algebraic.)

Questions

7. Using the following figure and questions 3 and 4, show that the side (e.g. BC) and the diagonal (e.g. BD) of a regular pentagon are incommensurable.



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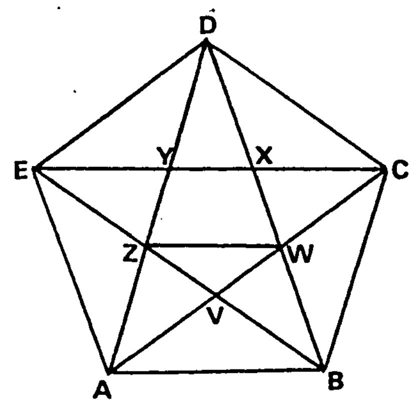
* See, for example

Fritz, K. von, The discovery of incommensurability by Hippasus of Metapontum. *Annals of Mathematics*, 1945, 46 (6), p. 242-64.

3. In the case of the incommensurability of the diagonal and side of a square, $\sqrt{2}$ is involved. The irrationality of which number is implied in the case of the pentagon?

We answer this question in two stages

a)



We take ZV (side of the smaller pentagon) as of unit length. Denote ZW (diagonal of the smaller pentagon) by a. Hence

AZ = (Δ AZW is isosceles)

ZY =

YD =

ED =

AD =

b) Since ABCDE and VWXYZ are regular pentagons, the ratio of the diagonal to the side is the same in both; that is,

$$\frac{AD}{ED} = \frac{ZW}{ZV} .$$

Use part a) to find the answer to our question.

THE WEIZMANN INSTITUTE OF SCIENCE
REHOVOT · ISRAEL
DEPARTMENT OF SCIENCE TEACHING

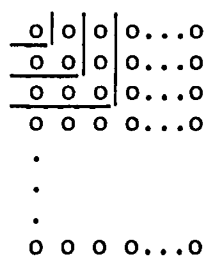
The Irrational Numbers

Answer sheet: The Pythagoreans

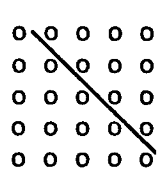
From the Pythagorean world

Figurate numbers

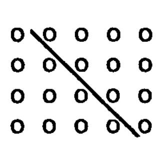
1. a) A square number (n^2) id equal to the sum of all the odd numbers which precede n.



b) The sum of two consecutive triangular numbers is a square number.



c) The sum of two equal triangular numbers is rectangular.



2. a) We can use the formula for the sum of an arithmetic series:

$$S_n = na_1 + \frac{(n-1)\cdot n\cdot d}{2} \,,$$

where, the first term $a_1 = 1$,

the difference between consecutive odd numbers $d = 2$.

Whence $S_n = n + (n-1)n = n^2$.

b) The representation

```
                            o
             o              o o          . . . .
o            o o            o o o
```

shows that the nth triangular number is

$$a_n = 1 + 2 + 3 + \ldots + n \,.$$

That is $a_n = \frac{n(n+1)}{2}$

Hence $\frac{n(n+1)}{2}$ and $\frac{(n+1)(n+2)}{2}$ are two consecutive triangular numbers. Their sum is

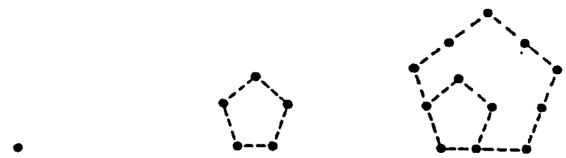
$$\frac{(n+1)}{2}(n+n+2) = \frac{(n+1)(2n+2)}{2} = (n+1)^2$$

c) From the previous section

$$2\,a_n = n(n+1)$$

which is triangular.

There are more (plane) figurate numbers, pentagonal, hexagonal, etc. For example, the sequence of pentagonal numbers is 1, 5, 12, 22,...



1 5 12

I.e. 1 1 + 4 1+4+7

$$a_n = 1 + 4 + \ldots + (3n - 2) = \frac{3n^2 - n}{2} .$$

The following are general rules for the construction of any (plane) figurate number.

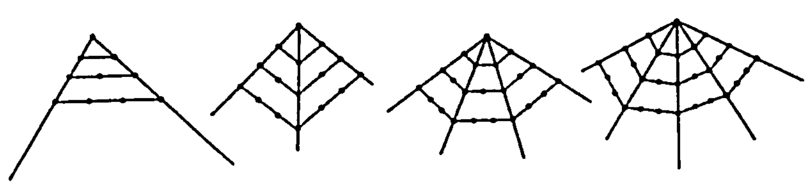
a) The number 1 is figurate for any type of figure.

b) Construct the second figurate number of the chosen type.

c) Choose one vertex of the polygon in b) and extend (in the direction of the adjacent vertices) the two sides meeting at that vertex by one point.

d) Construct the regular polygon of the chosen type on the two extended sides obtained in c).

An alternative method for constructing figurate numbers is illustrated in the following diagrams.



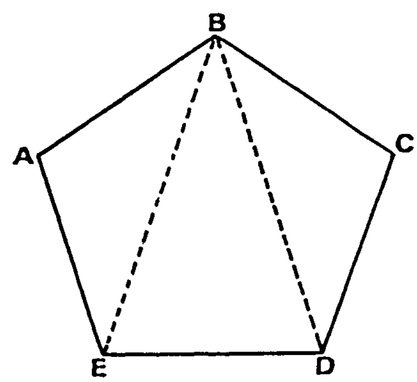
There are also 3-dimensional figurate numbers. For example, cubic numbers 1, 8, 27, ...

For further reading on figurate numbers see, for example,

- Bunt L.N.H., Jones P.S. and Bedient J.D., *The Historical Roots of Elementary Mathematics*, Prentice Hall, 1976, p.75ff.
- Boyer C.B., *A History of Mathematics*, J.Wiley & Sons, 1968, p. 59.

The pentagon

3. a)



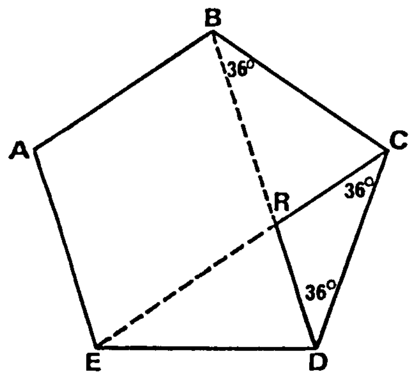
There are various ways of proving the required results. We shall use two theorems which were, apparently, known to the Pythagoreans*.

I - In an isosceles triangle the base angles are equal. And, conversely, if there are two equal angles in a triangle, then it is isosceles.

II - The sum of the (internal) angles of a triangle is two right angles (180^0).

If we divide the pentagon into three non-overlapping triangles (e.g. by the diagonals BD and BE), then it follows from II that the sum of the internal angles of the pentagon is 540^0. Since the pentagon is regular, each angle is 108^0. Further Δ BCD is isosceles, so the base angles are equal and each of them is 36^0.

* Fritz R. von, The discovery of incommensurability by Hippasus of Metapontum, *Annals of Mathematics*, 1945, 46, (6), p. 242-64.



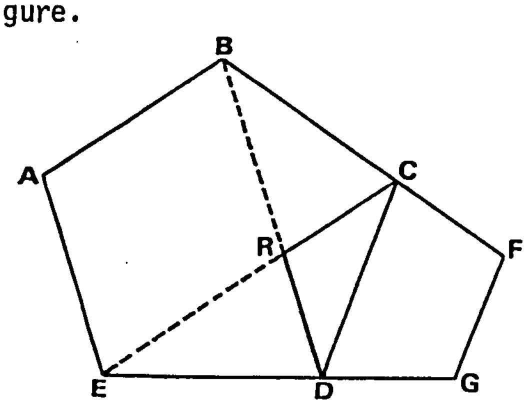
For the same reason (Δ CDE) angle ECD is also 36^0. Hence Δ RCD is isosceles.

b) From the angle sum in ΔCRD, we obtain

$$DRC = 108^0 = \sphericalangle A \ .$$

c) $BCR = 108^0 - 36^0 = 72^0$, and from the angle sum in ΔBCR, $\sphericalangle$ BRC is also equal to 72^0. Hence ΔBCR is isosceles, i.e. BC = BR .

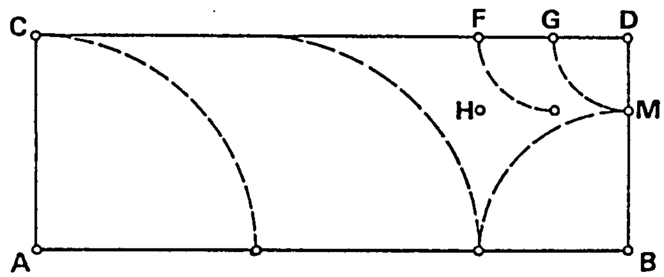
Since RC = RD and $\sphericalangle$ CRD = 108^0, we can construct a second regular pentagon RCFGD as shown in the following figure.



We shall use this construction in Question 7.

Common measure

4.



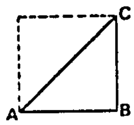
The segment FG (or FH) is the common measure. Since FD is equal to MB, we have

$$AC = 3a \quad \text{and} \quad AB = 8a \ .$$

Hence the ratio of the segment AC to the segment AB is as 3 to 8.

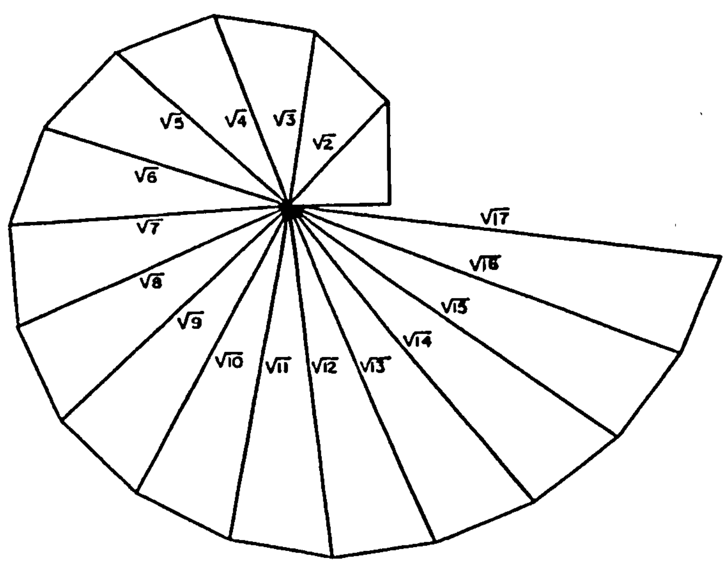
The discovery of incommensurable segments

5.

Data	Required to prove	Proof	Reason
The square:  AB = BC $AB^2 + BC^2 = AC^2$ That is $2AB^2 = AC^2$	The diagonal and side of the square are segments without a common measure. That is integers m and n do not exist, such that $\frac{AC}{AB} = \frac{m}{n}$	Either $\frac{AC}{AB} = \frac{m}{n}$ or $\frac{AC}{AB} \neq \frac{m}{n}$ Suppose $\frac{AC}{AB} = \frac{m}{n}$ m and n are integers with no common factor. →	For if they had a common factor, we could always cancel and obtain m and n without a common factor.
		⇓ $\frac{AC^2}{AB^2} = 2$ →	See data
		⇓ $\frac{m^2}{n^2} = 2$ →	By assumption
		⇓ $m^2 = 2n^2$	
		⇓ m^2 is even 2 →	m is either even or odd. Assume that it is odd. Then m = 2k + 1 for some $k \in N$. Hence $m^2 = 4k^2 + 4k + 1$ which is odd, and hence contradicts the previous conclusion that m^2 is even.
		⇓ m is even ⇓ m = 2r ⇓ $m^2 = 4r^2$ ⇓ $2n^2 = 4r^2$ ⇓ $n^2 = 2r^2$ ⇓ n^2 is even ⇓ n is even ⇓ n and m have a common factor which contradicts our assumption above.	

6. In his book *Theatetus*, Plato states that Theodorus proved that the sides of squares whose area is 3,5,..., 17 do not have a common measure with the side of the square of unit area. Unfortunately he does not say how. Why did he stop at 17? This question has interested many historians of Greek mathematics.

   i) - iii) There is a theory (brought by Heath*, but he rejects it) that the meaning of "proved" is no more than that he demonstrated by construction, that such segments actually exist. The required construction is shown in the following figure.



We see that the next step, $\sqrt{18}$, overlaps the first. This could be one of the possible reasons for Theodorus' stopping at 17.

---

* Heath T., *A History of Greek Mathematics*, Vol I, Oxford, 1921, p. 203-7.

iv) Hardy* suggests that it is possible that Theodorus, dealt with each case separately. In which case, to prove that the side of a square of area 3 does not have a common measure with the side of a unit square, one has to show that

$$\frac{m^2}{n^2} \neq 3 .$$

The proof is similar to that in Question 5, except that in this case, we have to show that

$$m^2 \text{ a multiple of } 3 \Longrightarrow m \text{ a multiple of } 3 .$$

As before, assume that m is not a multiple of 3, then either

$$m = 3k - 1 \quad \text{or} \quad m = 3k + 1 \text{ , } k \varepsilon N$$

Hence

$$m^2 = 9k^2 - 6k + 1 = 3(3k^2 - 2k) + 1 \quad \text{or} \quad m^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$$

In either case m^2 is of the form $3\ell + 1$, $\ell \varepsilon N$, and hence cannot be a multiple of 3, which contradicts our previous assumption.
Hence we conclude that m is a multiple of 3, and the remainder of the proof is exactly as in Question 5.

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* Hardy G.H. & Wright E.M., *An Introduction to the Theory of Numbers*, Oxford, 1945, p. 42-5.

The proof for the square of area 5 is again similar. This time we have to show that

$$m^2 \text{ a multiple of } 5 \implies m \text{ a multiple of } 5$$

Assume that, on the contrary,

$$m^2 \text{ a multiple of } 5 \implies m \text{ \underline{not} a multiple of } 5,$$

then m must be of the form

$$m = 5k + r , \quad k \epsilon N , \quad r = 1, 2, 3, 4.$$

Hence $m^2 = 5k^2 + 10kr + r^2$

$$= 5(5k^2 + 2kr) + r^2$$

Since r^2 is equal to one of 1, 4, 9, 16, none of which are multiples of 5. Hence, once again, the contradiction.

If we attempt a similar proof for the square of area 8, a new point appears. As before, we have to show that

$$m^2 \text{ a multiple of } 8 \implies m \text{ a multiple of } 8$$

Assume that, on the contrary,

$$m^2 \text{ a multiple of } 8 \implies m \text{ \underline{not} a multiple of } 8$$

then m must be of the form

$$m = 8k + r , \quad k \epsilon N , \quad r = 1, 2, \ldots, 7 .$$

Hence $m^2 = 64k^2 + 16kr + r^2$

$$= 8(8k^2 + 2kr) + r^2 .$$

Only this time r^2 can be equal to any one of 1, 4, 9, 16, 25, 36, 49, and if $r^2 = 16$, we do not get a contradiction with our assumption that m^2 is a multiple of 8. This does not mean that a contradiction does not exist, only that this method does not lead us to one.

In fact, we go right back to the beginning of the argument, we want to show that

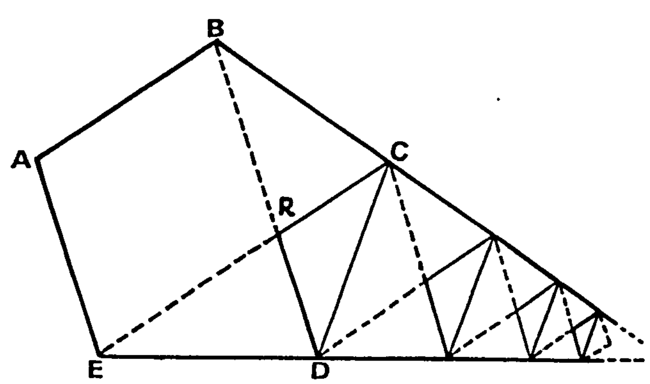
$\frac{m^2}{n^2} \neq 8$. Assume that $\frac{m^2}{n^2} = 8$, i.e.

$$m^2 = 8n^2$$

Then $$m^2 = 2(2n)^2$$

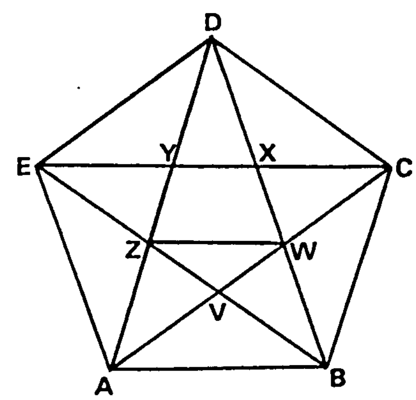
and this is exactly the same situation as in the proof of the square of area 2 .

7.



In this case, to show that BC and BD do not have a common measure, we use the method of mutual subtraction, described earlier (pages 4 - 6) in the worksheet. Since BC = BR (Question 3), the problem is equivalent to finding the common measure of BR and BD. Hence, the next step is to find the common measure of BR and RD. But BR = CD, so that the problem is reduced to finding the common measure of RD and CD, which are, once again, the side and diagonal of a regular pentagon. Since this process can be continued indefinitely, we conclude that the side and diagonal of a regular pentagon do not have a common measure.

8. a)



AZ = a , since Δ AZW is isosceles

ZY = 1 , side of small pentagon

YD = a , Δ EZA is congruent to Δ EYD

ED = 1 + a , ED = DY + YZ (Qu. 3)

AD = 1 + 2a .

b) $$\frac{1 + 2a}{1 + a} = \frac{a}{1}$$

That is $a^2 - a - 1 = 0$

This equation has one positive root $\frac{\sqrt{5} + 1}{2}$.

This number is known as the *golden section* and denoted by the Greek letter ϕ (phi).

It has many interesting properties, some of which are included in a book on the subject, *De Divina Proportione,* by the Italian Luca Pacioli, published in 1509.

The following are a few of its properties:

- The golden section is the only positive number such that the difference between it and its multiplicative inverse is 1.
- The following are two infinite representations of the golden section.

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}} \qquad \sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}$$

- The Fibonacci sequence (named after the Italian thirteenth century mathematician, Leonardo of Pisa, who was also known by the pseudonym, Fibonacci), is defined as follows:

$$a_1 = 1$$

$$a_2 = 1$$

$$a_n = a_{n-1} + a_{n-2}$$

That is, each term in the sequence is obtained as the sum of the two preceding terms, the first two terms being 1. Thus, the first few terms of the sequence are

1, 1, 2, 3, 5, 8, 13, 21, ...

The ratio $\frac{a_n}{a_{n-1}}$, as n tends to infinity, tends to the golden section.

- The designation "section" arose because of the occurrence of the number in a geometrical context:-

  If a line segment AB be divided at a point D, such that the ratio of the whole (AB) to the larger of the sub-segments (AD, say), is equal to the ratio of AD to DB, then this ratio is again Ø .

  Notice that, in the figure of the pentagon in Question 8, Y divides AD in this ratio, i.e.

  $\frac{AD}{AY} = \frac{AY}{YD}$. Similarly Z divides AD in the same ratio.

  Further Y divides ZD (and Z divides AY) in the same ratio.

  Further details on the golden section can be found, for example, in

Gardner M., *The 2nd Scientific American Book of Mathematical Puzzles and Diversions*, Simon & Schuster N.Y., 1961, p. 89-103.

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The Irrational Numbers

Worksheet: Euclid and *The Elements*

Incommensurables

Very little is known about Euclid. He lived about the year 300 B.C.E. and apparently taught in Alexandria. He is famous as the author of the *Elements,* arguably the best known book in the history of mathematics. Since the invention of printing there have been numerous editions in many languages - and many adaptations. A standard English edition is that of Heath*, who added a scholarly commentary. The following extracts are to set the scene. The first extract is taken from Heath's introduction.

---

* Heath T.L., *The Thirteen Books of Euclid's Elements Translated with Introduction and Commentary.* Dover Pub., N.Y., 1956.

> As in the case of the other great mathematicians of Greece, so in Euclid's case, we have only the most meagre particulars of the life and personality of the man.
> Most of what we have is contained in the passage of Proclus' summary relating to him, which is as follows : *
>
> . . . Euclid, who put together the Elements, collecting many of Eudoxus' theorems, perfecting many of Theaetetus', and also bringing to irrefragable demonstration the things which were only somewhat loosely proved by his predecessors . . . **
>
> . . . It is most probable that Euclid received his mathematical training in Athens from the pupils of Plato; for most of the geometers who could have taught him were of that school, and it was in Athens that the older writers of elements, and the other mathematicians on whose works Euclid's *Elements* depend, had lived and taught.

From Heath***

> This wonderful book, with all its imperfections, which indeed are slight enough when account is taken of the date at which it appeared, is and will doubtless remain the greatest mathematical text-book of all time. . . .

---

* Proclus (410-485 C.E.) is one of the commentators through whom many details of Greek mathematics have been preserved.

** To Eudoxus and Theaetetus are attributed many ideas which are brought in the *Elements*, in particular those in Book X, from which we quote later.

*** Heath T.L., *Greek Mathematics*, Vol. I, Oxford Univ. Press, 1921, p. 357-358.

From Boyer*

The *Elements* is divided into thirteen books or chapters, of which the first half dozen are on elementary plane geometry, the next three on the theory of numbers, Book X on incommensurables, and the last three chiefly on solid geometry. There is no introduction or preamble to the work, and the first book opens abruptly with a list of twenty-three definitions. The weakness here is that some of the definitions do not define, inasmuch as there is no prior set of undefined elements in terms of which to define the others. Thus to say, as does Euclid, that "a point is that which has no part," or that "a line is breadthless length," or that "a surface is that which has length and breadth only," is scarcely to define these entities, for a definition must be expressed in terms of things that precede, and are better known than the things defined. ...

Following the definitions, Euclid lists five postulates and five common notions. Aristotle had made a sharp distinction between axioms (or common notions) and postulates; the former, he said, must be convincing in themselves -- truths common to all studies - but the latter are less obvious and do not presuppose the assent of the learner, for they pertain only to the subject at hand ...

... In its time the *Elements* evidently was the most tightly reasoned logical development of elementary mathematics that had ever been put together, and two thousand years were to pass before a more careful presentation occurred. During this long interval most mathematicians regarded the treatment as logically satisfying and pedagogically sound.

---

* Boyer C.B., *A History of Mathematics*, J. Wiley & S., 1968, p. 115-118.

Question

1. The extract from Boyer distinguishes between definitions, postulates and common notions (axioms). Decide which of these three is the appropriate description for each of of the following statements.*

   a) There exists only one straight line containing two given points.

   b) The whole is greater than the part.

   c) Things which are equal to the same thing are also equal to one another.

   d) All right angles are equal to one another.

   e) An obtuse angle is an angle greater than a right angle.

---

* The sentences are taken from Heath's English version of the *Elements*.

The following extract (from Heath's edition of the *Elements*), contains examples of definitions taken from Book X, dealing with incommensurables.

DEFINITIONS.

1. Those magnitudes are said to be **commensurable** which are measured by the same measure, and those **incommensurable** which cannot have any common measure.

2. Straight lines are **commensurable in square** when the squares on them are measured by the same area, and **incommensurable in square** when the squares on them cannot possibly have any area as a common measure.

3. With these hypotheses, it is proved that there exist straight lines infinite in multitude which are commensurable and incommensurable respectively, some in length only, and others in square also, with an assigned straight line. Let then the assigned straight line be called **rational**, and those straight lines which are commensurable with it, whether in length and in square or in square only, **rational**, but those which are incommensurable with it **irrational**.

Questions

2. Give an example of line-segments which are commensurable in square only.

3. Which of the following statements are correct and which not? Give examples (or counter examples) for each statement.

   a) Segments which are commensurable in length are commensurable in square.

   b) Segments which are incommensurable in square are incommensurable in length.

   c) Segments which are commensurable in square are commensurable in length.

   d) Segments which are incommensurable in length are incommensurable in square.

4. Given the following four squares

Square	Area
ABCD	1
EFGH	2
KLMN	3
PQRS	4

   We choose the diagonal of the square PQRS as the "assigned straight line (rational)". Which of the following segments are rational and which irrational?

   a) The diagonal of the square EFGH.

   b) The diagonal of the square KLMN.

   c) The diagonal of the square ABCD.

d) The side MN.

e) The side PQ.

f) The segment formed by the combination of the side KL and the side PQ.

In each case, determine whether the segment is commensurable in length and/or in square, with the given segment.

5. Draw a Venn diagram to represent the relation between rational and irrational segments with respect to a given segment and their commensurability.

6. What is the difference between the concept "rational" ("irrational") as defined by Euclid, and that we use today.

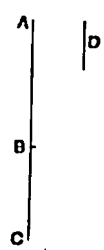
The following is a further extract taken from Heath's edition of Book X.

PROPOSITION 16.

If two incommensurable magnitudes be added together, the whole will also be incommensurable with each of them; and, if the whole be incommensurable with one of them, the original magnitudes will also be incommensurable.

For let the two incommensurable magnitudes AB, BC be added together;

I say that the whole AC is also incommensurable with each of the magnitudes AB, BC.



For, if CA, AB are not incommensurable, some magnitude will measure them.

Let it measure them, if possible, and let it be D.

Since then D measures CA, AB,

therefore it will also measure the remainder BC.

But it measures AB also;

therefore D measures AB, BC.

Therefore AB, BC are commensurable;

but they were also, by hypothesis, incommensurable:

which is impossible.

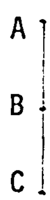
Therefore no magnitude will measure CA, AB;

therefore CA, AB are incommensurable. [x. Def. 1]

Similarly we can prove that AC, CB are also incommensurable.

Question

7. Proposition 16 contains two statements and the proof of one of them. The proof is given in the following table. Complete it.

Data	To prove	Deduction	In modern algebraic notation
	1) AC and AB are incommensurable	1) There are two possibilities: AB and AC are commensurable or they are not	
Such that AB and BC are incommensurable magnitudes.		Suppose the former...	- - - - - -
		- - - - -	
		- - - - -	
		⇓	
		- - - - -	- - - - - -
		⇓	
		- - - - - - -	- - - - - - -

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The Irrational Numbers

Answer sheet: Euclid and the *Elements*.

1. a) and d) are postulates, since they are related to the subject matter.

   b) and c) are axioms - "truths common to all studies". (Note, however, that whereas c) might be described to be universal, b) is not. Thus, in modern times, an infinite set was defined as a set which can be put into one-one correspondence with a subset of itself. For example,

   1, 2, 3, 4, 5, ..., n, ...

   2, 4, 6, 8, 10, ...,2n, ...

   And, in this sense, the "whole is not greater than its part".)
   Finally e) is a definition of a new term.
   (Notice also that d) is a postulate logically linked to the axiom in c).)

   In the first book of the *Elements*, Euclid brings 23 definitions (point, line, circle, etc.), 5 postulates and 5 axioms.
   In the following we bring the 5 postulates as they appear in the Boyer* version.

---

* See
Boyer C.B., *A History of Mathematics*, Wiley, 1968, p.116-7.

Postulates.

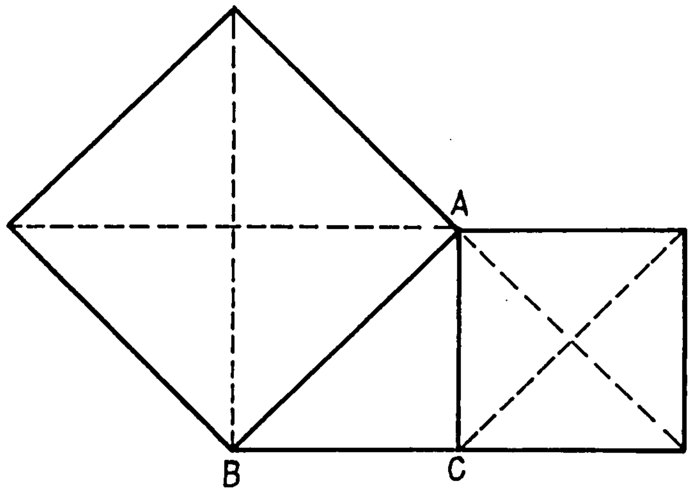
1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any center and radius.
4. That all right angles are equal.
5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than the two right angles.

The fifth postulate is distinguished from the rest by its complicated enunciation, and it was the most problematic of the five. For some 2000 years, mathematicians tried to prove it and/or to propose alternatives (Heath lists nine), such as

- through a given point, there is just one straight line parallel to a given straight line.
- if a straight line cuts one of two parallel straight lines, then it also cuts the second.

In the end, in the nineteenth century there was an interesting and surprising development. The mathematicians Gauss (1777-1855), Bolyai (1802-1860) and Lobachevskii (1792-1856) showed that it was possible to construct a consistent geometry, and not only one, in which the fifth postulate was negated. Such geometries are called *non-Euclidean*. The existence of such geometries shows that postulate 5 cannot be proved.

2. The simplest example is the diagonal and side of a square. The following figure shows that they are commensurable in square and incommensurable in length (as was shown in the previous worksheet).



The square on AC and the square on AB have a common measure; for example, triangle ABC measures both of them.

3. a) This statement is correct, since if two given lengths have a length which is their common measure, then one can construct a square on it which will measure the squares on the given lengths.

   b) This statement is also correct, since if there is no square which is the common measure of two given squares, there cannot be a length which measures the sides of the two squares.

   In fact, a) and b) are logically equivalent. For if the first is of the form

$$p \Rightarrow q ,$$

   then the second is of the form

$$\sim q \Rightarrow \sim p ,$$

   where $\sim p$ denotes the negation of p ("not p"). And for any two statements p, q much that $p \Rightarrow q$, we have $\sim q \Rightarrow \sim p$; i.e.

$$(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$$

c) This is not true, as the example given in the solution of of question 2 shows.

d) This is also not true and again the example in question 2 is sufficient to show this. This is not surprising since c) and d) are logically equivalent.

4. The various lengths and areas are given in the following table.

Square	Area	Side	Diagonal
ABCD	1	1	$\sqrt{2}$
EFGH	2	$\sqrt{2}$	2
KMNL	3	$\sqrt{3}$	$\sqrt{6}$
PQRS	4	2	$\sqrt{8}$

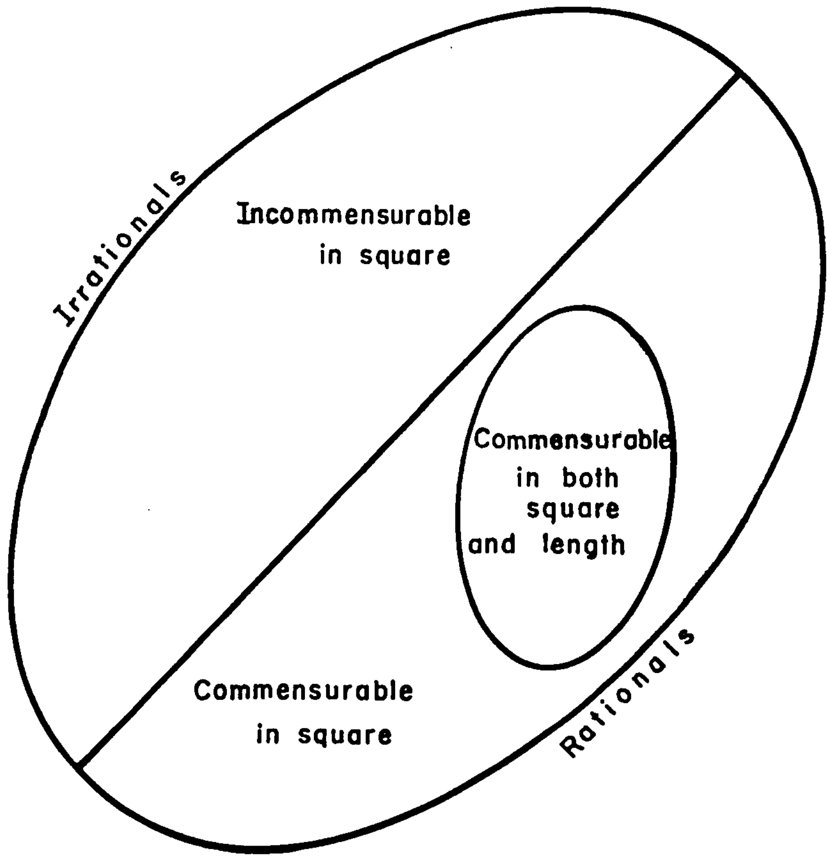
where $\sqrt{8}$ is the "assigned straight line (rational)".

Hence

Length	Rational	Irrational	Commensurable in length	Commensurable in square
a) diagonal EFGH	x	-	-	x
b) diagonal KLMN	x	-	-	x
c) diagonal ABCD	x	-	x	x
d) side MN	x	-	-	x
e) side PQ	x	-	-	x
g) KL + PQ	-	x	-	-

Note that to be rational, a length has only to be commensurable with the given length either in square or in length, and not necessarily both.

5.



Notice that

commensurable in square $\Longleftrightarrow$ rational

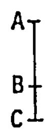
incommensurable in square $\Longleftrightarrow$ irrational ,

whereas this is not true if we replace "square" by "length" (see question 3).

6. Euclid's use of rational and irrational is relative, depending on the choice of "given length", whereas today

it is an absolute classification of numbers. Further, even if we take, for example, the given length to be the unit (1), Euclid's rational includes, among other things, $\sqrt{2}$ because it is commensurable in square with 1.

7.

Data	To prove	Deduction	In modern algebraic notation
 Such that AB and BC are incommensurable magnitudes, i.e. there is no measure  D which measures them both	1) AC and AB are incommensurable magnitudes 2) BC and AC are incommensurable magnitudes	1) There are two possibilities: AB and AC are commensurable or they are not.	
		Suppose the former and that D is the common measure →	Suppose AC=mD AB=nD, where m and n are natural numbers.
		⇓ D will also measure the remainder BC* →	$BC = AC - AB$ $= (m - n)D$
		⇓ AB and BC are commensurable, which contradicts our assumption ⇓ AC and AB are incommensurable.	
		2) Similarly.	

---

* This was proved in the previous theorem (theorem 15).

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The Irrational Numbers

Worksheet: Irrationals in the 16th and 17th century

In this worksheet we bring extracts* from mathematical texts written in the 16th and 17th century, in which various mathematicians discuss the irrationals.

a) Michael Stifel (1487-1567) was a German mathematician. The mathematical historian Boyer** maintains that his work was of great significance in Germany in the 16th century. The work in question is Stifel's book *Arithmetica Integra* (1544).

The following extract, which is taken from Kline's book***, refers to Stifel's position with regard to irrationals, as expressed in an extract which he brings from the original (translated into English).

---

* The extracts are quoted as they appear in other texts. since we had no access to the primary source. Therefore, we note from where the extract is taken and who quotes it.

** Boyer C.B., *A History of Mathematics,* Wiley, 1968, p. 309-310.

*** Kline M., *Mathematical Thought from Ancient to Modern Times*. Oxford U.P., 1972, p. 251.

In his major work, the *Arithmetica Integra* (1544), which deals with arithmetic, the irrationals in the tenth book of Euclid, and algebra, Stifel considers expressing irrationals in the decimal notation. On the one hand, he argues:

> Since, in proving geometrical figures, when rational numbers fail us irrational numbers take their place and prove exactly those things which rational numbers could not prove ... we are moved and compelled to assert that they truly are numbers, compelled that is, by the results which follow from their use—results which we perceive to be real, certain, and constant. On the other hand, other considerations compel us to deny that irrational numbers are numbers at all. To wit, when we seek to subject them to numeration [decimal representation] ... we find that they flee away perpetually, so that not one of them can be apprehended precisely in itself.... Now that cannot be called a true number which is of such a nature that it lacks precision.... Therefore, just as an infinite number is not a number, so an irrational number is not a true number, but lies hidden in a kind of cloud of infinity.

He then argues that real numbers are either whole numbers or fractions; obviously irrationals are neither, and so are not real numbers.

The following is the last few lines of the above, but in the original Latin of the *Arithmetica Integra*, as brought by Tropfke*.

"Sicut igitur infinitus numerus non est numerus, sic irrationalis numerus non est verus numerus, et latet sub quadam infinitatis nebula...Deinde, si numeri irrationalis essent numeri veri, tunc aut essent integri aut essent fracti...

---

* Tropfke J., *Geschichte der Elementarmathematik*, vol. I Verlag Von Veit & Co., Leipzig 1902, p. 161.

b) Simon Stevin (1548-1620) was an engineer from the Netherlands. He made many important contributions in the field of statics and hydrostatics, which were particularly relevant in the Netherlands. But he also secured himself a place in the history of mathematics, because of his development of decimal fractions. The following short extract is taken from his book *Arithmetique* (1585), as it is quoted by Klein*.

**

"La partie est de la mesme matiere que son entier; Racine de 8 est partie de son quarré 8: Doncques $\sqrt{8}$ est de la mesme matiere que 8: Mais la matiere de 8 est nombre; Doncques la matiere de $\sqrt{8}$ est nombre: Et par consequent $\sqrt{8}$ est nombre."

---

* Klein J., *Greek Mathematical Thought and the Origin of Algebra*. M.I.T. Press, 1968, p. 290.

** This picture was taken from
Beck A., Bleicher M.N. & Crowe D.W., *Excursions into Mathematics*. Worth Pub. Inc., 1976, p. 446.

c) Albert Girard (1595-1632) was a student of Stevin and also from the Netherlands. Girard published Stevin's books, and his works include *Invention Nouvelle en L'Algebre* (1629). The following extract is taken from this book as quoted by Klein*.

> "Notez qu'on appelle *un nombre* tant les radicaux simples, comme est $\sqrt{2}$, ou $\sqrt{5071}$, que les multinomes, commes les binomes $2+\sqrt{5}$, item $7-\sqrt{48}$, item $\sqrt{26}-5$, comme les trinomes $4+\sqrt{2}-\sqrt{17}$, et autres multinomes, car ce qui lié par les signes soit+soit—ne font qu'un nombre."

Questions

1. What do you think Stifel meant when he wrote "when rational numbers fail us"?
2. What are Stifel's arguments for and against the acceptance of irrational numbers, and what is his conclusion?
3. Discuss Stevin's approach to roots and Girard's extension of this.
4. What does Girard mean by the term *multinome*, and what does this term mean today? What would we call Girard's *multinome*?
5. Could one of Girard's *multinomes* be a rational number?

---

* Ibid.

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The Irrational Numbers

Answer sheet: Irrationals in the 16th and 17th century.

1. It would seem reasonable to assume that Stifel was referring to finding the diagonal, given the side of the square. Or it could be the problem of finding the length of the side, given the area of a square. Etc.

2. Stifel's arguments are

For the acceptance of irrationals	against
- In geometry "irrational numbers... prove exactly those things which rational numbers could not prove". - "... the results which follow from their use-results which we perceive to be real, certain and constant."	- "they flee away perpetually", "...lies hidden in a kind of cloud of infinity"- meaning that they cannot be expressed

The mathematical historian Tropfke* adds that Stifel gives the irrational numbers a place in the sequence of numbers. Tropfke quotes the following extract from the *Arithmetica Integra* in the original Latin and with a German translation.

---

* Tropfke, J., *Geschichte der Elementarmathematik*, Vol. 1, Von Veit and Co., Leipzig 1902, p. 166-167.

> *Item licet infiniti numeri fracti cadant inter quoslibet duos numeros immediatos, quemadmodum etiam infiniti numeri irrationales cadunt inter duos numeros integros immediatos.* Ebenso wie die unbestimmten Brüche zwischen zwei benachbarte ganze Zahlen fallen, so fallen auch die unbestimmten Irrationalzahlen zwischen zwei benachbarte ganze Zahlen.

In spite of this, Stifel seems to conclude that "an irrational number is not a true number". Kline* states that this conclusion did not prevent Stifel from dealing with irrational numbers.

There were other contemporary mathematicians who worried about irrational numbers. Among them was J. Peletier (1517-1582), who wrote a chapter** entitled "Are the irrational numbers numbers or not, and of what kind?" The following extract is taken from the book by Klein,*** which gives the original Latin text with English translation.

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* Kline, M., *Mathematical Thought from Ancient to Modern Times*. O.U.P., 1972, p. 251.

** In a Latin text - *De Occulta Parte numerorum, quam Algebram Vocant (1560)*.

*** Klein, J., *Greek Mathematical Thought and the Origin of Algebra*, M.I.T. Press, 1968, p. 290.

"id tanquam in perpetuis tenebris delitescit" (this must lie hidden, as it were, in perpetual darkness). They are, in any case, "something," and it is certain that their *use is necessary* (necessarium usum), especially "in laying off continuous magnitudes" (praesertim in Continuorum dimensionibus). They are, just like genuine "absolute" numbers, subject to "rules" (praeceptiones). Their relation to the "absolute" numbers may be compared with that of the animals to man: "Habent igitur numeri Irrationales cum Absolutis obscuram quandam mutemque communicationem, non secus quam cum hominibus, Bruta: quae praeter id quod sentiunt, suo etiam modo ratiocinantur." (Irrational numbers, then, have a certain obscure and mute communication with absolute numbers, not differently from that which brutes, who besides having sense impressions, do, in their own way, even reason, have with men.) All in all, they are inexplicable (inexplicabiles) and have only a kind of shadow existence. They must not be counted among the numbers,

Peletier's arguments in favour of accepting the irrationals are similar to those of Stifel - "their use is necessary" - but he also is not willing to give them equal status with other numbers.

3. In the extracts, Stevin regards a root "as part of" its square in a sort of physical analogy - as if it were, for example a piece of metal. Every part of that metal is, of course, also metal - so a part of a number is a number. Whereas, Stifel took the mathematical view of a number as a sequence of digits, and got into difficulty with an infinite sequence, Stevin sidesteps the issue by using an implied analogy.

   Girard states that roots are numbers - probably having accepted the arguments of his predecessors - and adds that arithmetic expressions obtained by adding or subtracting numbers are also numbers.

Note that the extracts are extremely short, and we should not come to very definite conclusions without examining them more fully in context. This, however, is not relevant here, where all we are trying to do is give an impression of the reaction to irrational numbers in various periods leading up to their final acceptance and definition.

4. Girard's *multinomes* are numerical (or arithmetical) expressions of the form $2 + \sqrt{5}$ or $4 + \sqrt{2} - \sqrt{17}$. *Multinome* today is not used - but would seem to be equivalent to polynome, which is an algebraic expression involving the addition (or subtraction) of a number of terms, usually positive integer powers of one or more variables.

5. Yes. For example $\sqrt{2} - \sqrt{2} = 0$, or

$$2 + \sqrt{27} - \sqrt{12} - \sqrt{3} = 2$$

But, if we take two different rationals, a and b, such that $\sqrt{a}$ and $\sqrt{b}$ are irrational , we can prove that $\sqrt{a} \pm \sqrt{b}$ is irrational.

For, suppose that $\sqrt{a} \pm \sqrt{b}$ is rational, i.e. that

$$\sqrt{a} \pm \sqrt{b} = \frac{m}{n}$$

Then $$\sqrt{a} = \frac{m}{n} \mp \sqrt{b}$$

Hence $$a = \left(\frac{m}{n}\right)^2 \mp \frac{2m}{n}\sqrt{b} + b$$

or $$\sqrt{b} = \mp \frac{n}{2m}\left[a - b - \left(\frac{m}{n}\right)^2 \right]$$

Now the right-hand side of this equation is rational, which contradicts the fact that $\sqrt{b}$ is irrational.

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The Irrational Numbers

Worksheet: Rafael Bombelli

L'ALGEBRA
OPERA

Di RAFAEL BOMBELLI da Bologna
Diuiſa in tre Libri.

Con la quale ciaſcuno da ſe potrà venire in perfetta cognitione della teorica dell'Arimetica.

Con vna Tauola copioſa delle materie, che in eſſa ſi contengono.

Poſta hora in luce à beneficio delli ſtudioſi di detta profeſſione.

IN BOLOGNA,
Per Giouanni Roſsi. MDLXXIX.
Con licenza de' Superiori.

A method of approximating roots of natural numbers

One of the problems which interested mathematicians was, how to approximate an irrational number which occurred as the root of a natural number which is not a perfect square. In this worksheet we will learn about a method for approximating roots, as presented by Bombelli in his book *L'Algebra Opera* (1579). Bombelli lived in the sixteenth century in Bologna. Very little about him is given in the history books*. However most of the histories of mathematics regard his book highly, as one of the most advanced of its time. This judgement was apparently based mainly on the first three parts, since the fourth and fifth part was first published in 1929, from an apparently unknown manuscript.

The following extract is taken from pp. 35-37 of the 1579 edition. We did not find one English translation to the whole original extract which is the object of this work-sheet. Nevertheless we include:

- the English translation to the first paragraph, by V. Sanford (in Smith D.E., *A Source Book in Mathematics,* McGraw Hill, N.Y., 1929, p. 80-82)
- the English translation of the last paragraph as it appears in Dedron, P. & Ytard, J. *Mathematics and Mathematicians,* Vol. 2, Transworld Pub., 1974, p. 69-70.

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* Details can be found, for example, in

Jayawardene S.A., Rafael Bombelli, Engineer Architect: Some Unpublished Documents..., *Isis*, 56, 1965, p. 298-306.

Modo di formare il rotto nella estrattione delle Radici quadrate.

Molti modi sono stati scritti da gli altri autori de l'vso di formare il rotto; l'uno tassando, e accusando l'altro (al mio giudicio) senza alcun proposito, perche tutti mirano ad un fine; E ben vero che l'una è più breue dell'altra, ma basta che tutte supplisçono, e quella ch'è più facile, non è dubbio ch'essa sarà accettata da gli huomini, e sarà posta in uso senza tassare alcuno; perche potria essere, che hoggi io insegnassi una regola, laquale piacerebbe più dell'altre date per il passato, e poi venisse un'altro, e ne trouasse una più vaga, e facile, e così sarebbe all'hora quella accetata, e la mia confutata, perche (come si dice) la esperienza ci è maestra, e l'opra loda l'artefice. Però metterò quella che più à me piace per hora, e sarà in arbitrio de gli huomini pigliare qual vorranno: dunque venendo al fatto dico. Che presuposto, che si voglia il prossimo lato di 13, che sarà 3, e auanzerà 4, il quale si partirà per 6 (doppio del 3 sudetto) ne uiene $\frac{2}{3}$, e questo è il primo rotto, che si hà da giongere al 3, che fà 3 $\frac{2}{3}$, ch'è il prossimo lato di 13, perche il suo quadrato è 13 $\frac{4}{9}$, ch'è superfluo $\frac{4}{9}$, ma uolendosi più approssimare, al 6. doppio del 3 se gli aggionga il rotto, cioè li $\frac{2}{3}$, e farà 6 $\frac{2}{3}$, e per esso partendosi il 4, che auanza dal 9 fino al 13, ne uiene $\frac{3}{5}$, e questo si gionge al 3, che fà 3 $\frac{3}{5}$, ch'è il lato prossimo di 13, di cui il quadrato è 12 $\frac{24}{25}$, ch'è più prossimo di 3 $\frac{2}{3}$, ma uolendo più prossimo, si aggionga il rotto al 6 fa 6 $\frac{3}{5}$, e con esso si parta pur il 4, ne uiene $\frac{20}{33}$, e questo si aggionga, come si è fatto di sopra al 3 fà 3 $\frac{20}{33}$, ch'è l'altro numero più prossimo, perche il suo quadrato è 13 $\frac{4}{1089}$, ch'è troppo $\frac{4}{1089}$, e uolendo piu prossimo, partasi 4 per 6 $\frac{20}{33}$, ne uiene $\frac{132}{218}$, che gionto con il 3 fà 3 $\frac{132}{218}$, e questo è piu prossimo del passato, che il suo quadrato è 13 [illegible] e così procedendo si può approssimare à una cosa insensibile.

Method of Forming Fractions in the Extraction of Roots

Many methods of forming fractions have been given in the works of other authors; the one attacking and accusing another without due cause (in my opinion) for they are all looking to the same end. It is indeed true that one method may be briefer than another, but it is enough that all are at hand and the one that is the most easy will without doubt be accepted by men and be put in use without casting aspersions on another method. Thus it may happen that today I may teach a rule which may be more acceptable than those given in the past, but if another should be discovered later and if one of them should be found to be more vague and if another should be found to be more easy, this [latter] would then be accepted at once and mine would be discarded; for as the saying goes, experience is our master and the result praises the workman. In short, I shall set forth the method which is the most pleasing to me today and it will rest in men's judgment to appraise what they see: meanwhile I shall continue my discourse going now to the discussion itself.

Let us first assume that if we wish to find the approximate root[1] of 13 that this will be 3 with 4 left over. This remainder should be divided by 6 (double the 3 given above) which gives $\frac{2}{3}$. This is the first fraction which is to be added to the 3, making $3\frac{2}{3}$ which is the approximate root of 13. Since the square of this number is $13\frac{4}{9}$, it is $\frac{4}{9}$ too large, and if one wishes a closer approximation, the 6 which is the double of the 3 should be added to the fraction $\frac{2}{3}$, giving $6\frac{2}{3}$, and this number should be divided into the 4 which is the difference between 13 and 9. The result is $\frac{3}{5}$ which, added to the 3 makes $3\frac{3}{5}$. This is a closer approximation to the root of 13, for its square is $12\frac{24}{25}$, which is closer than that of the $3\frac{2}{3}$. But if I wish a closer approximation, I add this fraction to the 6 making $6\frac{3}{5}$, divide 4 by this, obtaining $\frac{20}{33}$. This should be added to the 3 as was done above, making $3\frac{20}{33}$. This is a closer approximation for its square is $13\frac{4}{1089}$, which is $\frac{4}{1089}$ too large. If I wish a closer approximation, I divide 4 by $6\frac{20}{33}$, obtaining $\frac{66}{109}$, [and] add this to 3, obtaining $3\frac{66}{109}$. This is much closer than before for its square is $13\frac{96}{11881}$, which is $\frac{96}{11881}$ too large. If I wish to continue this even further, I divide 4 by $6\frac{66}{109}$ obtaining $\frac{109}{180}$, [the author now continues the process and then remarks:] and this process may be carried to within an imperceptible difference.

... epro cedendo (come si è fatto di sopra) si approssimarà quanto l'huomo vorrà, e se bene ci sono molte altre regole: queste nondimeno mi sono parse le più facili, però a queste mi atterrò, le quali hò trouato con fondamento, qual non uoglio restare di porlo, benche non sarà inteso, se non da chi intende l'agguagliare di potenze, e tanti eguali à numeri, del quale tratterò nel secondo libro à pieno: Però hora parlo solo con quelli.

Pongasi dunque, che si habbia à trouare il lato prossimo di 13, di cui il più prossimo quadrato è 9; di cui il lato è 3, però pongo che il lato prossimo di 13. sia 3. p. 1 tanto, e il suo quadrato è 9. piu 6 tanti p. 1. potenza, ilqual'è eguale à 13. che leuato 9. a ciascuna delle parti, resta 4, eguale à 6 tanti più 1 potenza. Molti hanno lasciato andare quella potenza, e solo hanno agguagliato 6 tanti à 4, che il tanto valeria $\frac{2}{3}$ & hanno fatto, che l'approssimatione si è 3 $\frac{2}{3}$ perche la positione fù 3.p.1. tanto, uiene ad essere 3 $\frac{2}{3}$, ma uolendo tenere conto della potenza ancora, valendo il tanto $\frac{2}{3}$, la potenza ualerà $\frac{2}{3}$ di tanto, che aggionto con li 6 tanti di prima: si hauerà 6 $\frac{2}{3}$ tanti eguale à 4, che agguagliato il tanto valerà $\frac{3}{5}$, e perche fù posto 3. p. 1. tanto, sarà 3 $\frac{3}{5}$, e ualendo il tanto $\frac{3}{5}$, la potenza valerà $\frac{3}{5}$ di tanto, e si hauerà 6 $\frac{3}{5}$ di tanto eguale à 4, si che si uede donde nascono le regole dette di sopra.

Let us suppose we are required to find the root of 13. The nearest square is 9, which has root 3. I let the approximate root of 13 be 3 plus 1 tanto. Its square is 9 plus 6 tanti p. 1 power. We set this equal to 13. Subtracting 9 from either side of the equation we are left with 4 equal to 6 tanti plus 1 power.

Many people have neglected the power and merely set 6 tanti equal to 4. The tanto then comes to $\frac{2}{3}$ and the approximate value of the root is $3\frac{2}{3}$ since it has been set equal to 3 p. 1 tanto. However, taking the power into account, if the tanto is equal to $\frac{2}{3}$, the power will be $\frac{2}{3}$ of a tanto, which, added to the 6 tanti, will give us $6\frac{2}{3}$ tanti, which are equal to 4. So the tanto will be equal to $\frac{3}{5}$, and since the approximate root is 3 p. 1 tanto it comes to $3\frac{3}{5}$. But if the tanto is equal to $\frac{3}{5}$, the power will be $\frac{3}{5}$ of a tanto and we obtain $6\frac{3}{5}$ tanti equal to 4....

Questions

1. At the end of the extract, Bombelli gives the principles of his method for *"chi intende l'agguagliare di potenze, e tanti equali à numeri"* (those who understand how to equalize powers and unknowns numbers).

   Complete the following "dictionary".

Bombelli	English translation	Modern notation
p. or *piu*	plus	------
equale	equal	------
I tanto	one quantity (unknown)	------
potenza	power (second, of unknown)	------
3 p. I tanto	------	3 + x
9. piu 6 tanti p. I potenza	------	------

2. The stages in Bombelli's method of approximating $\sqrt{13}$ are set out in the following table. Complete the blanks

Bombelli (original)	English version	Modern Notation	The method
...pongo che il lato proſſimo di 13. ſia 3. p. 1 tanto, e il ſuo quadrato è 9. piu 6 tanti p. 1. potenza, ilqual'è eguale à 13. che leuato 9. a ciaſcuna delle parti, reſta 4, eguale à 6 tanti più 1 potenza. Molti hanno laſciato andare quella potenza, e ſolo hanno agguagliato 6 tanti à 4, che il tanto valeria $\frac{2}{3}$ & hanno fatto, che l'approſſimatione ſi è $3\frac{2}{3}$...	I let the approximate root of 13 be 3 plus 1 tanto. Its square is 9 plus 6 tanti p. 1 power. We set this equal to 13. Subtracting 9 from either side of the equation we are left with 4 equal to 6 tantj plus 1 power. Many people have neglected the power and merely set 6 tanti equal to 4. The tanto then comes to $\frac{2}{3}$ and the approximate value of the root is $3\frac{2}{3}$...	$(3+x)^2=9+6x+x^2=13$ ---------- ---------- ---------- ----------	1st. approx. x_1: $6x_1=4$ $x_1=\frac{4}{6}$
...ma uolendo tenere conto della potenza ancora, valendo il tanto $\frac{2}{3}$, la potenza ualerà $\frac{2}{3}$ di tanto, che aggionto con li 6 tanti di prima: ſi hauerà $6\frac{2}{3}$ tanti eguale à 4, che agguagliato il tanto valerà $\frac{3}{5}$...	However, taking the power into account, if the tanto is equal to $\frac{2}{3}$, the power will be $\frac{2}{3}$ of a tanto, which, added to the 6 tanti, will give us $6\frac{2}{3}$ tanti, which are equal to 4. So the tanto will be equal to $\frac{3}{5}$...	$6x+x^2=4$ $6x+xx=4$ ------ ------	2nd approx. x_2: $6x_2+\frac{4}{6}x_2=4$ ---------
...e ſi hauerà $6\frac{2}{3}$ di tanto eguale à 4, ...	...and we obtain $6\frac{2}{3}$ tanti equal to 4....	------	3rd approx. x_3: ---------- ----------

3. In the section where Bombelli approximates $\sqrt{13}$ (on page 2), there is a mistake. Find it! What is its probable cause? (This question refers to the original Italian because the mistake does not appear in the English version.)

4. Approximate $\sqrt{2}$ by Bombelli's method, accurate to 4 decimal places (1.4142).

5. a) If b is not a perfect square, write down the iteration formula which gives the approximation to $\sqrt{b}$ by Bombelli's method.

   b) Is the first approximation to $\sqrt{b}$, greater than or less than $\sqrt{b}$?

   c) What about the second and third approximation?

   d) Generalise !

6. Bombelli writes "*... e cosi procedendo si puo approssimare a una cosa insensibile*" (... and this process may be carried to within an imperceptible difference), but in the extract, he "omits" the fundamental mathematical question, whether, if we continue the process indefinitely, one can prove that the sequence of approximations converges to the desired root. Do it in two stages:

   a) Show that the odd approximations are steadily decreasing and the even ones are increasing. (Remember also what you discovered in the previous question.)

   b) Show that the odd and the even approximations converge to the desired root.

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The Irrational Numbers

Answer sheet: Rafael Bombelli

A method for approximating roots of natural numbers

Bombelli	English translation	Modern notation
p. or *piu*	plus	+
equale	equal	=
I tanto	one quantity (unknown)	x
potenza	power (second, of unknown)	x^2
3 p. I tanto	3 plus one quantity	$3 + x$
9 piu 6 tanti p. I potenza	9 plus 6 quantities plus one power	$9 + 6x + x^2$

Bombelli (original)	English version	Modern Notation	The method
... pongo che il lato proſſimo di 13. ſia 3. p. 1 tanto, e il ſuo quadrato è 9. piu 6 tanti p. 1. potenza, ilqual'è eguale à 13. che leuato 9. a ciaſcuna delle parti, resta 4, eguale à 6 tanti più 1 potenza. Molti hanno laſciato andare quella potenza, e ſolo hanno agguagliato 6 tanti à 4, che il tanto valeria ÷ & hanno fatto, che l'approſſimatione ſi è 3 ÷ ...	I let the approximate root of 13 be 3 plus 1 tanto. Its square is 9 plus 6 tanti p. 1 power. We set this equal to 13. Subtracting 9 from either side of the equation we are left with 4 equal to 6 tantj plus 1 power. Many people have neglected the power and merely set 6 tanti equal to 4. The tanto then comes to $\frac{2}{3}$ and the approximate value of the root is $3\frac{2}{3}$ ...	$(3+x)^2=9+6x+x^2=13$ $9+6x+x^2-9=13-9$ $6x+x^2=4$ ⋮ $6x=4$ $x=\frac{2}{3}$ $3+\frac{2}{3}=3\frac{2}{3}$	1st. approx. x_1: $6x_1=4$ $x_1=\frac{4}{6}$
... ma uolendo tenere conto della potenza ancora, valendo il tanto ÷, la potenza ualerà ÷ di tanto, che aggionto con li 6 tanti di prima: ſi hauerà 6 ÷ tanti eguale à 4, che agguagliato il tanto valerà ÷ ...	However, taking the power into account, if the tanto is equal to $\frac{2}{3}$, the power will be $\frac{2}{3}$ of a tanto, which, added to the 6 tanti, will give us $6\frac{2}{3}$ tanti, which are equal to 4. So the tanto will be equal to $\frac{3}{5}$...	$6x+x^2=4$ $6x+xx=4$ $6x+\frac{2}{3}x=4$ $6\frac{2}{3}x=4$ $x=\frac{3}{5}$	2nd approx. x_2: $6x_2+\frac{4}{6}x_2=4$ $x_2=\dfrac{4}{6+\frac{4}{6}}$
... e ſi hauerà 6 ÷ di tanto eguale à 4, ...	... and we obtain $6\frac{3}{5}$ tanti equal to 4....	$6\frac{3}{5}x=4$	3rd approx. x_3: $6x_3+\dfrac{4}{6+\frac{4}{6}}x_3=4$ $x_3=\dfrac{4}{6+\dfrac{4}{6+\frac{4}{6}}}$

Thus, we can write the general procedure in the form

$$\sqrt{13} = 3 + \cfrac{4}{6 + \cfrac{4}{6 + \cfrac{4}{6 + \cdots}}}$$

This expression is today known as an *infinite continued fraction* - not that Bombelli wrote it this way or called it by this name. Nevertheless, he is credited with being one of the first to develop the subject, which has an interesting history of its own.

3. After the approximation $3\frac{20}{33}$, Bombelli writes:

> uolendo piu proſſimo, partaſi 4 per 6 [illegible], ne uiene [illegible], che gionto con il 3 fà 3 [illegible].

but $4 \div 6\frac{20}{33} = 4 \times \frac{33}{218} = \frac{66}{109}$,

and not $\frac{109}{180}$, as appears in the original.

If we continue the process, we get

$$4 \div 6\frac{66}{109} = 4 \times \frac{109}{720} = \frac{109}{180}.$$

So, it would seem that the error occurred as a result of skipping one step, maybe caused by missing out a line from the manuscript when setting the type. (It is interesting to note that in Smith* this section is given in English translation, for the same Italian edition as we have used here, without any error.)

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* Smith, D.E. *A Source Book in Mathematics*, McGraw-Hill, 1929, p. 80-82.

4. $$(1 + x)^2 = 1 + 2x + x^2 = 2$$

$$2x + x^2 = 1$$

If we ignore the x^2, we obtain

$$x_1 = \frac{1}{2}$$

Hence the first approximation is $1\frac{1}{2}$, whose square is $2\frac{1}{4}$.

Continuing the procedure, we obtain

$$2x_2 + \frac{1}{2}\,x_2 = 1$$

$$2\tfrac{1}{2}\,x_2 = 1$$

$$x_2 = \frac{1}{2\frac{1}{2}} = \frac{2}{5}$$

Hence, the second approximation is $1\frac{2}{5}$, whose square is $1\frac{24}{25}$.

At the next stage

$$2x_3 + \frac{2}{5}\,x_3 = 1$$

$$x_3 = \frac{1}{2\frac{2}{5}} = \frac{5}{12}$$

Hence the third approximation is $1\frac{5}{12}$ $(1.41\dot{6})$

Etc.

5. a) Let a be the largest natural number whose square is less than b, that is

$$a^2 < b < (a + 1)^2.$$

Hence, there exists an x, $0 < x < 1$, such that

$$(a + x)^2 = b$$

$$a^2 + 2ax + x^2 = b$$

$$2ax + x^2 = b - a^2$$

For the first approximation, we ignore x^2 (since $x < 1$, x^2 is small in comparison with 2ax) and obtain

$$2ax_1 = b - a^2$$

$$x_1 = \frac{b - a^2}{2a}$$

For the second approximation, we write

$$2ax_2 + x_1x_2 = b - a^2$$

$$(2a + x_1)x_2 = b - a^2$$

$$x_2 = \frac{b - a^2}{2a+x_1}$$

In general

$$x_{n+1} = \frac{b - a^2}{2a + x_n}$$

The nth approximation to $\sqrt{b}$ is then $a + x_n$.

b) From the previous section, the first approximation to $\sqrt{b}$ is $a + x_1$.

$$(a + x_1)^2 = a^2 + 2ax_1 + x_1^2$$

But since $a^2 + 2ax_1 = b$, we have

$$(a + x_1)^2 = b + x_1^2 > b \ .$$

Thus the first approximation is greater than $\sqrt{b}$.

c)

$$x_2 = \frac{b - a^2}{2a + x_1}$$

Hence $(2a + x_1)x_2 = b - a^2$

$$[(a + x_1) + a]x_2 = (\sqrt{b} - a)(\sqrt{b} + a)$$

But $a + x_1$ is the first approximation, and we have shown in the previous section that $a + x_1 > \sqrt{b}$. Hence replacing $a + x_1$, by $\sqrt{b}$, we have

$$(\sqrt{b} + a)x_2 < (\sqrt{b} - a)(\sqrt{b} + a)$$

i.e. $x_2 < \sqrt{b} - a$

or $a + x_2 < \sqrt{b}$

showing that the second approximation is less than $\sqrt{b}$.

We can show similarly that the third approximation is again greater than $\sqrt{b}$.

d) Exactly the same sort of argument will show that every even approximation is always less than b , and every odd approximation always greater. In symbols

$$a + x_{2n} < \sqrt{b} \qquad n \varepsilon N$$
$$a + x_{2n-1} > \sqrt{b}$$

6. a) We have to show that for $n \varepsilon N$

$$x_{2n-1} > x_{2n+1}$$

or $$x_{2n-1} - x_{2n+1} > 0 .$$

Using the iteration formula we found in the previous question, we have

$$x_{2n+1} = \frac{b - a^2}{2a + x_{2n}}$$

$$x_{2n+1} = \frac{b - a^2}{2a + \frac{b - a^2}{2a + x_{2n-1}}}$$

$$x_{2n+1} = \frac{(b-a^2)\cdot(2a+x_{2n-1})}{2a\cdot(2a+x_{2n-1}) + b - a^2}$$

hence

$$x_{2n-1} - x_{2n+1} = x_{2n-1} - \frac{(b-a^2)\cdot(2a+x_{2n-1})}{2a(2a+x_{2n-1}) + b - a^2}$$

$$= \frac{x_{2n-1}[2a(2a+x_{2n-1}) + b - a^2] - (2a + x_{2n-1})\cdot(b-a^2)}{2a(2a+x_{2n-1}) + b - a^2}$$

Since the denominator is positive, we need only examine the numerator, which becomes

$$x_{2n-1} \cdot 2a \cdot (2a+x_{2n-1}) - 2a(b-a^2)$$

$$= 2a[x_{2n-1}(2a+x_{2n-1}) - (b-a^2)]$$

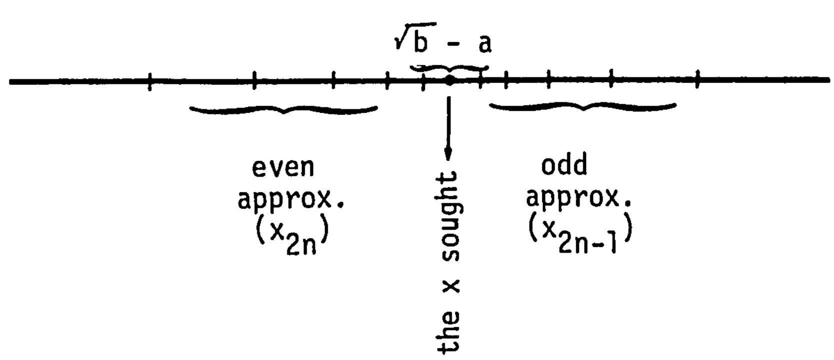
$$= 2a[x_{2n-1}^2 + 2ax_{2n-1} + a^2 - b]$$

$$= 2a[(x_{2n-1}+a)^2 - b] .$$

Since odd approximations are greater than $\sqrt{b}$, the expression in brackets is positive; i.e. $x_{2n-1} > x_{2n+1}$

Similarly, it can be shown that each even approximation is greater than its predecessor.

b) From what we have proved so far, the situation is described by the following graph



The odd approximations are decreasing but are always greater than $\sqrt{b} - a$. Therefore, they must approach some limit, ℓ_1 say, such that

$$\ell_1 > \sqrt{b} - a$$

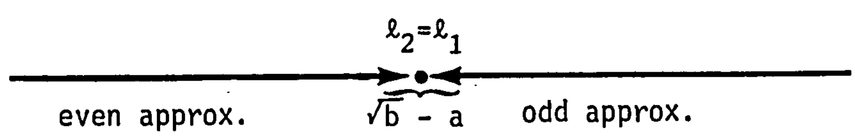
Similarly the even approximations approach some limit ℓ_2 , such that

$$\ell_2 < \sqrt{b} - a$$

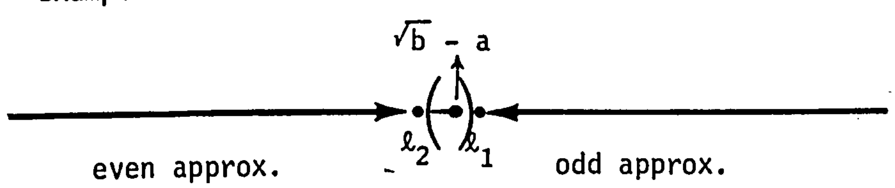
It remains to show that

$$\ell_1 = \ell_2 = \sqrt{b} - a \ .$$

That is



and that we do not create a situation such as, for example



Now, we have

$$x_n = \frac{b - a^2}{2a + x_{n-1}} = \cfrac{b - a^2}{2a + \cfrac{b - a^2}{2a + x_{n-2}}}$$

If n is even (odd), then n-2 is even (odd). As n increases, x_n and x_{n-2} are very close to ℓ_2 (ℓ_1). Hence

$$\ell_1 = \frac{b - a^2}{2a + \frac{b - a^2}{2a + \ell_1}}$$

$$\ell_1 = \frac{(b - a^2)\cdot(2a+\ell_1)}{2a(2a+\ell_1) + b - a^2}$$

$$[2a(2a+\ell_1) + b - a^2]\ell_1 = (b-a^2)\cdot(2a+\ell_1)$$

$$2a\ell_1(2a+\ell_1) = 2a(b-a^2)$$

$$\ell_1^2 + 2a\ell_1 + a^2 = b$$

$$(\ell_1 + a)^2 = b$$

whence $$\ell_1 = x$$

Similarly one can show that

$$\ell_2 = x .$$

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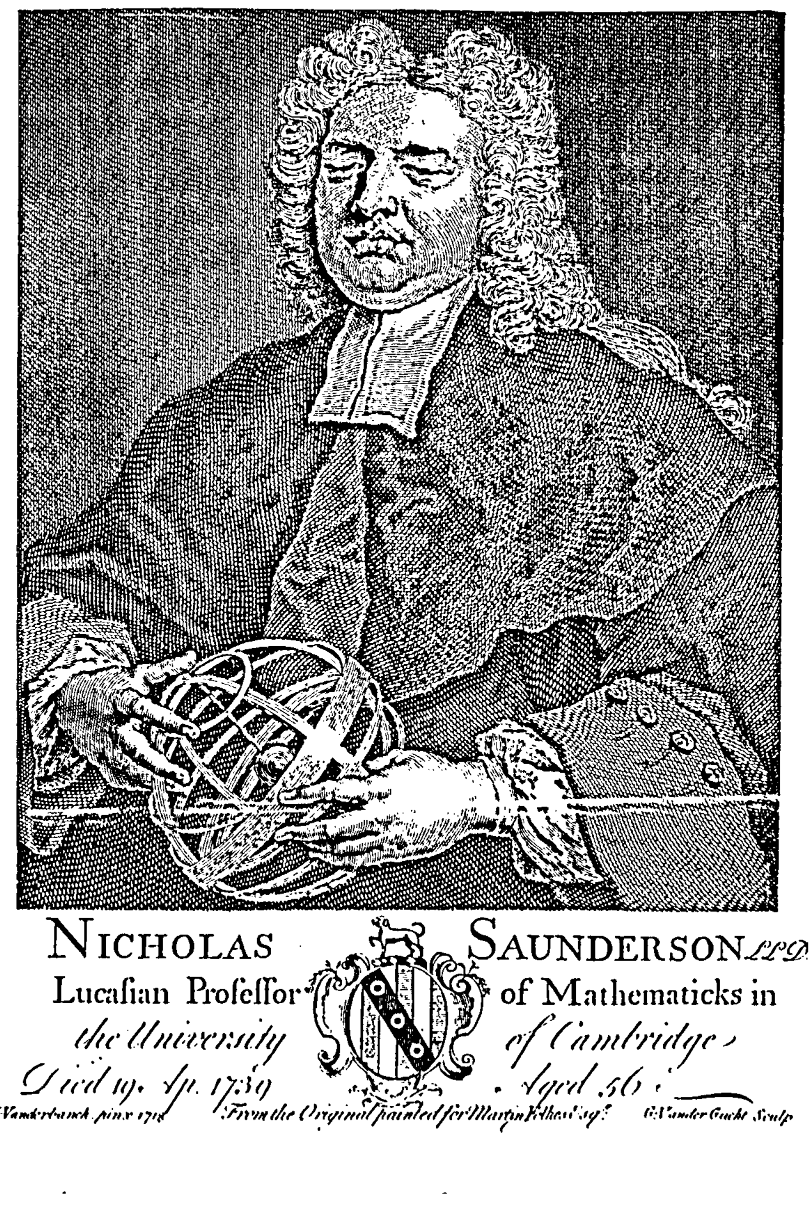
The Irrational Numbers

Worksheet: Nicholas Saunderson



Calculating the square root

Nicholas Saunderson (1682-1739), an English mathematician, became blind at the age of one as a result of smallpox. This did not prevent him from learning, teaching and creating mathematics. As a young man he learnt Greek, Latin and French, and hence "heard" the works of Euclid, Archimedes, Diophantes and others in the original, and remembered parts of them by heart.

His father taught him arithmetic and he showed considerable ability in solving problems and long calculations, using his prodiguous memory and a "geoboard", which he developed. The "geoboard" was a pegboard with arrays of nails which represented numbers. By passing silk threads around the nails, he could "draw" geometrical figures.

Saunderson continued his mathematical studies with great success, having texts read to him. He liked teaching mathematics, and gave very clear expositions. At Cambridge he had the distinction of occupying the chair held by Newton some years earlier.

The following extracts are taken from his book The *Elements of Algebra** (pages 165-9), which was published posthumously in 1741. In the extracts he gives a mathematical explanation of the "usual rule" for finding square roots.

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* Further details on the man and his work can be found in the introduction to the book itself,
Saunderson N., *The Elements of Algebra*, Vol 1. Cambridge Univ. Press, 1741.

For the better effecting what I here propoſe, I ſhall lay down the following obſervations, which the learner muſt attend to, if he expects to go through the following demonſtration; and if there be ſtill any difficulties he may meet with or thinks he meets with in the application of theſe obſervations, the beſt advice I can give him here, as well as in many other parts of this book, is to read the demonſtration over and over again, by which means all the ſteps will become more familiar to him, and he will be the better able to put them together, in order to digeſt the whole.

OBSERVATION 1.

If any number, as $a+x$, conſiſting of two parts a and x, be ſquared, the product will be $aa+2ax+xx=aa+\overline{2a+x}\times x$.

OBSERVATION 2.

$1\times 1=1$; therefore $10\times 10=100$, and $100\times 100=10000$, and $1000\times 1000=1000000$, &c: whence I infer, that if any number conſiſts of one or two places, it's ſquare root, or at leaſt the integral part of it, will conſiſt but of one place; if a number conſiſts of three or four places, the integral part of it's ſquare root will conſiſt of two places...

OBSERVATION 3.

From the laſt obſervation it follows, that if of any number propoſed, a point be put over the place of units, and another over the next place but one to the left hand, I mean that of hundreds, and another again over the next place but one to that, to wit, of ten thouſands, and ſo on alternately; the number of points will diſcover the number of places of which the integral part of the ſquare root of the propoſed number conſiſts: thus if the number 56644 be ſo pointed, it will ſtand thus, $\dot{5}6\dot{6}4\dot{4}$; and the three points ſhew, that the ſquare root of this number, or the integral part of it at leaſt, conſiſts of three places.

OBSERVATION 4.

...if the ſquare root of the number 576 be 24, that of the number 57600 will be 240...

OBSERVATION 5.

...if the ſquare root of any number, ſuppoſe of 5 .6644 lies between 2 and 3, the ſquare root of 566 .44 will lie between 20 and 30...

The inveſtigation of the rule for extracting the ſquare root.

Theſe things being obſerved, let now ſome number be propoſed, ſuch as is the number 56644; and let us ſee whether we cannot inveſtigate the ſquare root of this number by pure dint of reaſon, without any regard to, or helps from the common method.* This number then being pointed according to the third obſervation, I ſhall firſt begin with the number 5, belonging to the firſt point to the left hand, ſetting aſide all the other figures, thus: the number 5 is itſelf no ſquare number, therefore I ſubtract the number 4 which is the neareſt ſquare number leſs than 5, from 5, and there remains 1...

...the number 2 will be the firſt term of the ſquare root,...

...now to diſcover the next figure in the root, I conſider the number 566 belonging to the two firſt dotts to the left hand, ſetting aſide all the other figures,...

...the ſquare root of the number 566 lies between 20 and 30; let then $20+x$ repreſent the integral part of the ſquare root of the number 566, the letter x ſtanding for ſome whole number in the place of units; then it is plain, that the ſquare of $20+x$ muſt either be preciſely equal to 566 if 566 be an exact ſquare, or elſe it muſt be the neareſt ſquare number leſs than 566; but the ſquare of $20+x$ by the firſt obſervation is $20\times20+\overline{40+x}\times x$; therefore $20\times20+\overline{40+x}\times x$ is either equal to, or leſs than 566; ſubtract the ſquare of 20 from both ſides, by obſerving, that as before, 2×2 ſubtracted from 5 left 1, ſo now 20×20 ſubtracted from 500 will leave 100, and the ſame ſubtracted from 566, will leave 166; therefore $\overline{40+x}\times x$ muſt either be equal to, or leſs than 166... ...it muſt be reſolved into two factors, x and $40+x$,...

---

* According to the introduction to this section (which we have not brought here), Saunderson means by the "common method", the method which he is about to explain. However, he asks the reader to forget the rules, in order to better understand their mathematical justification.

<u>Questions</u>

1. i) What is the significance of the line above $\overline{2a + x}$?

   ii) Write Observation 1 in modern form.

2. Generalise the second observation for the square root (or the integer part thereof) of an n-digit number.

3. Generalise the fourth observation for $a \cdot 10^n$, where $\sqrt{a} = b$.

4. Complete Saunderson's explanation of the finding of the square root of 56644.

5. Find $\sqrt{13}$, by the method explained in the extract, to three decimal places.

6. How does Saunderson's method differ from that of Bombelli?

7. Are you aware of any other methods for finding square roots?

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The Irrational Numbers

Answer sheet: Nicholas Saunderson

Calculating the square root

1. The line above $\overline{2a + x}$ is used where we today would use brackets. Thus we write $a^2 + (2a + x)x$ instead of $aa + \overline{2a + x} \times x$.

   Although the use of brackets to associate terms can be found in 16th century texts, according to Cajori*, they did not become widely used until the 18th century. Beside the line as used by Saunderson, terms to be associated were denoted by letters, commas or full stops. The line, which was called a *vinculum,* persisted in use, even after the general introduction of brackets. For example, the following expression is taken from a textbook edited in 1950**, in which both the *vinculum* and the brackets are used.

$$= \frac{1}{2}\left\{2an_2 + 2n_1n_2d - n_2(n_2 + 1)\,d\right\}$$

$$= \frac{1}{2}\left\{2n_1n_2d - (dn_2^2 - \overline{2a - d}\,.\,n_2)\right\}$$

---

* Cajori F., *A History of Mathematical Notations,* Vol. I
The Open Court Pub. Co. 1928, p. 384-400.

** Hall H.S. & Knight S.R., *Higher Algebra,* Macmillan & Co., London, 1950, p. 34.

In the end, it is likely that the brackets triumphed for reasons associated with the printing of mathematical texts, rather than mathematical convenience.

2. According to Observation 2, the integral part of the square root of each number between 1 and 100, consists of one digit, between 100 and 10,000 of two digits, etc. Thus if a number consists of n digits, then if

   n is <u>even</u>, the integral part of the square root will have $\frac{n}{2}$ digits,

   n is <u>odd</u>, the integral part of the square root will have $\frac{n+1}{2}$ digits.

3. If $\sqrt{a} = b$ and n is even, then $\sqrt{a \cdot 10^n} = b\ 10^{\frac{n}{2}}$.

4. In Saunderson, the continuation is as follows.

> ... the product of whose multiplication must either be equal to 166, or else it must be the nearest product of the kind less: but to proceed; as it has been shewn already that the product $\overline{40+x} \times x$ is not to exceed 166, it follows, that the product $40x$ must be less than 166, and consequently that x must be less than the quotient of 166 divided by 40, or of 16.6 divided by 4; but the quotient of 16.6 divided by 4 lies between the two whole numbers 4 and 5; therefore x is less than 5, and 4 is the greatest whole number that can be supposed equal to x; let us then suppose x equal to 4, and let us see what will be the consequence; now if $x=4$, we shall have $40+x=44$, and $\overline{40+x} \times x = \overline{44 \times 4} = 176$; therefore the supposition of $x=4$ was wrong, because $\overline{40+x} \times x$ ought not to exceed 166; let us then suppose in the next place that $x=3$; then we shall have $40+x=43$, and $\overline{40+x} \times x = 43 \times 3 = 129$, which is less than 166, and therefore not inconsistent with the foregoing conditions; and since x must be taken equal to the greatest whole number the foregoing conditions will admit of, it follows, that x must be equal to 3, and consequently that $20+x$,
>
> or

or the integral part of the ſquare root of the number 566 is 23; and ſince $\overline{40+x}\times x$ or 129, ſubtracted from the reſolvend 166 leaves 37, it follows, that 23×23 ſubtracted from the number 566 will alſo leave 37: again, as the ſquare of 23 is the neareſt ſquare number leſs than 566 which is no ſquare, the ſquare of 24 muſt be greater than 566, and conſequently cannot be leſs than 567; therefore the ſquare root of the number 566.44 muſt neceſſarily lie between 23 and 24; therefore by the fifth obſervation, the ſquare root of the number 56644 muſt neceſſarily lie between 230 and 240; which is another ſtep made in the approximation: let us now ſuppoſe $230+x$ to be the ſquare root, or the integral part of the ſquare root of the number 56644; then by the firſt obſervation we ſhall have $230\times 230+\overline{460+x}\times x$ either equal to, or leſs than 56644: ſubtract 230×230 from both ſides, by obſerving, that as before 23×23 ſubtracted from 566 left 37, ſo now 230×230 ſubtracted from 56600 will leave 3700; and the ſame ſubtracted from 56644 will leave 3744; therefore $\overline{460+x}\times x$ muſt not exceed 3744; therefore $460x$ muſt be leſs than 3744; therefore x muſt be leſs than the quotient of 3744 divided by 460; but the quotient of 3744 divided by 460, or (which is pretty much the ſame thing, eſpecially with reſpect to the integral part) the quotient of 374 divided by 46 lies between the two whole numbers 8 and 9; therefore 8 is the greateſt whole number that can be ſuppoſed equal to x: let us then ſuppoſe $x=8$, and we ſhall have $\overline{460+x}=468$, and $\overline{460+x}\times x=468\times 8=3744$, which is juſt conſiſtent with the abovementioned conditions, and the number 238 is the exact ſquare root of the number propoſed 56644. *Q. E. I.*

We expect that your answer was somewhat shorter.

In order to organize the calculation, we suggest the following form.

$\sqrt{\dot{5}\,6\,\dot{6}\,4\,\dot{4}}$	238
$-\ 4$	$\boxed{2}\cdot 2 = 4$
1 6 6	$(40 + x)\cdot x \leq 166 \qquad / x = 3$
$-\ 1\,2\,9$	$4\boxed{3}\cdot\boxed{3} = 129$
	$23\cdot 2 = 46$
3 7 4 4	$(460 + x)\cdot x \leq 3744 \quad / \quad x = 8$
$-\ 3\,7\,4\,4$	$46\boxed{8}\cdot\boxed{8} = 3744$
0 0 0 0	

5.

$\sqrt{1\,\dot{3}.0\,\dot{0}\,0\,\dot{0}\,0\,\dot{0}\ldots}$	3.605...
$-\ 9$	$\boxed{3}\cdot 2 = 6$
4 0 0	$(60 + x)\cdot x \leq 400 \qquad / x = 6$
$-\ 3\,9\,6$	$6\boxed{6}\cdot\boxed{6} = 396$
	$36\cdot 2 = 72$
4 0 0	$(720 + x)\cdot x \leq 400 \qquad / x = 0$
$-\ 0$	$720\boxed{0}\cdot\boxed{0} = 0$
	$360\cdot 2 = 720$
4 0 0 0 0	$(7200 + x)\cdot x \leq 40000 \ / \ x = 5$
$-\ 3\,6\,0\,2\,5$	$720\boxed{5}\cdot\boxed{5} = 36025$
3 9 7 5	
⋱	⋱

In the following passage, Saunderson describes how to deal with numbers which are not perfect squares.

There are but few ſquare numbers, or ſuch as will admit of an exact ſquare root, in compariſon of the reſt; and therefore, whenever a number is propoſed to have it's ſquare root extracted, the artiſt muſt firſt determine with himſelf, to how many decimal places it is proper the root ſhould be continued; and then by annexing decimal cyphers, if need be, to the right hand of the number propoſed, he muſt make twice as many decimal places there, as the root is to conſiſt of; after this, he muſt put a point over the place of units, and then paſſing by every other figure, he muſt point in like manner all the reſt, both to the right hand, and to the left: by this means, the number will be prepared,...

That the ſuppoſed ſquare ought to have twice as many decimal places as the root, is evident, both *à priori*, and *à poſteriori*: *à priori*, becauſe in extracting the ſquare root, two figures are brought down from the ſquare for every ſingle figure gained in the root; and *à poſteriori*, becauſe the root multiplied into itſelf is to produce the ſquare; and therefore, from the nature of multiplication, the ſquare ought to have twice as many decimal places as the root.

6.

a) The method is essentially to "find" the root of any number, including perfect squares	a) The method is essentially to find approximations to the square root of a whole number, given the largest integer whose square is less than the given number.
b) The values obtained in successive stages are always less than the desired square root (except in the case of a perfect square, in the final stage).	b) The values obtained in successive stages are alternately smaller and larger than the square root.
c) The digits found in the successive stages of the process, do not change in subsequent stages. For example,	c) The digits found in successive stages of the process, may change in subsequent stages. For example,
First approx. $\sqrt{2} = 1.4$ Second approx. $\sqrt{2} = 1.41$ Third approx $\sqrt{2} = 1.414$ Fourth approx. $\sqrt{2} = 1.4142$	First approx. $\sqrt{2} = 1.5$ Second approx. $\sqrt{2} = 1.4$ Third approx. $\sqrt{2} = 1.41\dot{6}$ Fourth approx. $\sqrt{2} = 1.413\ldots$.

The method described by Saunderson has a long history. According to Smith*, in the fourth century, Theon of Alexandria found roots by this method, as did the Arabs, Hindus

* Smith, D.E., *History of Mathematics,* Vol. II, Dover, 1958, pp. 144-9.

and other mathematicians in the Middle Ages and in more modern times. But it was only later that explanations were added to justify the method. The method was taught in schools until very modern times - in spite of tables and slide rules - finally to be displaced by the calculator.

7. a) See for example the following extract*.

General Method.

Graphic Illustration. Suppose the square ABCD represents the number, and that it is known that A*bcd*, representing say n^2, is very nearly equal to it, so that n is a sort of first guess to the square root of the number.

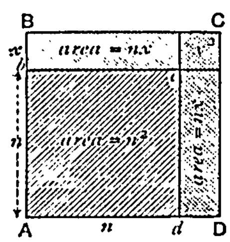

FIG. 37.

The problem is therefore to find the additional amount *b*B, representing say x, which has to be added to n to obtain the required square root.

From the figure it is clear that what is wanted is a number x, such that twice the product of x and n, together with the square of x, is equal to the difference between the square of n and the given number.

Further, that ABCD − A*bcd* is very nearly $2n \times x$, and therefore that x is *suggested by* (ABCD − A*bcd*) ÷ ($2n$).

This is the key of the Arithmetical Method of extracting square roots.

EXAMPLE.—*To evaluate* $\sqrt{4225}$.

Beginning with units and tens digits, divide up the number so far as possible into 'periods' of two digits.

60	42|25	5 = Units } portion of $\sqrt{4225}$; 60 = Tens } ∵ 4225 lies between 60^2 and 70^2.
	36 00	Set down 60 on both sides; multiply them as shown.
Twice 60 = 120	6 25	... = $4225 - 60^2$; $4225 - (60)^2 \div 120$ suggests 5 as units digit.
Add the 5 . 5		
125	6 25	Multiply 125 by 5.

Since there is no remainder $65 = \sqrt{4225}$.

---

* Taken from: Jones H.S., *A Modern Arithmetic with Graphic & Practical Exercises*, Macmillan & Co., 1909, p. 274-5.

b) With the advent of the calculator, a reasonable method for finding the square root of any number (without using the $\sqrt{\ }$ key) is trial and error. This can be formulated in the form of a systematic algorithm, and used quite generally.

c) Heron of Alexandria lived about the first century B.C.E. He is credited* with the following method and its explanation (which is given here in modern form).

If a_1 is a first approximation to $\sqrt{A}$, the Heron's second approximation is

$$a_2 = \frac{1}{2}\left(a_1 + \frac{A}{a_1}\right)$$

and, in general,

$$a_n = \frac{1}{2}\left(a_{n-1} + \frac{A}{a_{n-1}}\right)$$

This method also became known as Newton's (or Newton-Raphson) method, because it can be derived as a special case of a more general method for finding the roots of equations.

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* Dedron, P. and Itard, J., *Mathematics and Mathematicians*, Vol. 2, Transworld pub., 1973, pp. 67-8.

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The Irrational Numbers

Worksheet: Dedekind and the definition of the irrationals.

Introduction

1. What is the nearest rational to $\frac{1}{2}$?
2. How many rationals are there between 0.7 and 0.71?
3. Are all the numbers in the interval [0.7, 0.71] rational?
4. What is the number nearest to and greater than $\sqrt{2}$, and the nearest number smaller than it?

J.W.R. Dedekind

Julius Wilhelm Richard Dedekind (1831-1916) was a German mathematician. He developed a formal arithmetic definition of the irrational numbers, assuming the rationals as given. Dedekind was one of a number of distinguished German mathematicians who concerned themselves with this topic in the same period. Others included G. Cantor (1845-1918) and Weierstrass (1815-1897). The following extracts are taken from the English version of one of Dedekind's papers*. The original *Stetigkeit un die Irrationalzahlen* (Continuity and Irrational Numbers) was first published in 1872. He describes how the idea was born, and then gives the definition.

---

* Dedekind R., *Essays on the Theory of Numbers*. Open Court, 1924, pp. 1-24.

CONTINUITY AND IRRATIONAL NUMBERS.

MY attention was first directed toward the considerations which form the subject of this pamphlet in the autumn of 1858. As professor in the Polytechnic School in Zürich I found myself for the first time obliged to lecture upon the elements of the differential calculus and felt more keenly than ever before the lack of a really scientific foundation for arithmetic.

. . . .

The statement is so frequently made that the differential calculus deals with continuous magnitude, and yet an explanation of this continuity is nowhere given; even the most rigorous expositions of the differential calculus do not base their proofs upon continuity but, with more or less consciousness of the fact, they either appeal to geometric notions or those suggested by geometry, or depend upon theorems which are never established in a purely arithmetic manner.

Before presenting the arithmetic basis, Dedekind describes the process to be employed.

Just as negative and fractional rational numbers are formed by a new creation, and as the laws of operating with these numbers must and can be reduced to the laws of operating with positive integers, so we must endeavor completely to define irrational numbers by means of the rational numbers alone. The question only remains how to do this.

Question

5. Suggest a way to define the negative numbers in terms of the natural numbers, or the rational numbers in terms of integers.

With the central object of defining the irrationals, Dedekind first notes properties of the rational numbers.

I. If $a > b$, and $b > c$, then $a > c$.

II. If a, c are two different numbers, there are infinitely many different numbers lying between a, c.

III. If a is any definite number, then all numbers of the system R fall into two classes, A_1 and A_2, each of which contains infinitely many individuals; the first class A_1 comprises all numbers a_1 that are $< a$, the second class A_2 comprises all numbers a_2 that are $> a$; the number a itself may be assigned at pleasure to the first or second class, being respectively the greatest number of the first class or the least of the second. In every case the separation of the system R into the two classes A_1, A_2 is such that every number of the first class A_1 is less than every number of the second class A_2.

Dedekind then draws a parallel between these properties of the rational numbers and properties of points on a line L.

I. If p lies to the right of q, and q to the right of r, then p lies to the right of r; and we say that q lies between the points p and r.

II. If p, r are two different points, then there always exist infinitely many points that lie between p and r.

III. If p is a definite point in L, then all points in L fall into two classes, P_1, P_2, each of which contains infinitely many individuals; the first class P_1 contains all the points p_1, that lie to the left of p, and the second class P_2 contains all the points p_2 that lie to the right of p; the point p itself may be assigned at pleasure to the first or second class. In every case the separation of the straight line L into the two classes or portions P_1, P_2, is of such a character that every point of the first class P_1 lies to the left of every point of the second class P_2.

---

* Dedekind denotes the set of rational numbers by R

> This analogy between rational numbers and the points of a straight line, as is well known, becomes a real correspondence when we select upon the straight line a definite origin or zero-point o and a definite unit of length for the measurement of segments.

Questions

6. In property III (both for the rationals and the points on L), what sets are represented by

$$A_1 \cup A_2 \quad (\text{or } P_1 \cup P_2)$$

$$A_1 \cap A_2 \quad (\text{or } P_1 \cap P_2) \text{ ?}$$

7. Which of the following could be true? Explain. (In sections ii) to vii) $a_1 \varepsilon A_1$, $a_2 \varepsilon A_2$.)

   i) $A_1 = \phi$

   ii) $a_2 = a_1$

   iii) $a_1 < a_2$

   iv) $a_1 > a$

   v) $a_2 > a$

   vi) $a_1 = a$

   vii) $a_2 = a$

8. Prove that, if there exists a number $a_1 \varepsilon A_1$ which is greater than any other number in A_1, then there does not exists a number $a_2 \varepsilon A_2$, which is smaller than every other number in A_2.

Dedekind now continues as follows.

In the preceding section attention was called to the fact that every point p of the straight line produces a separation of the same into two portions such that every point of one portion lies to the left of every point of the other. I find the essence of continuity in the converse, i. e., in the following principle:

"If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions."

Question

9. Rewrite the axiom of continuity, given in terms of a line and the points which constitute it, in terms of *the system R* of rational numbers, and check whether you are willing to accept the latter also as an axiom.

The next stage in the process, as given by Dedekind, is:

CREATION OF IRRATIONAL NUMBERS.

From the last remarks it is sufficiently obvious how the discontinuous domain R of rational numbers may be rendered complete so as to form a continuous domain.

... any separation of the system R into two classes A_1, A_2, is given which possesses only *this* characteristic property that every number a_1 in A_1 is less than every number a_2 in A_2, then for brevity we shall call such a separation a *cut* [Schnitt] and designate it by (A_1, A_2). ...

But it is easy to show that there exist infinitely many cuts not produced by rational numbers.

Thus, the Dedekind cuts define the irrational numbers as follows.

- if (A_1, A_2) is such that A_1 has a greatest number a, (or A_2 has a least number a), then the cut is just another label for the rational number a.
- if no such a exists then the cut is a label for an irrational number - that is, the irrationals are now defined as "cuts" of the rationals.

10. Define $\sqrt{2}$.

11. Define π .

Having completed the definition of the irrationals (and with the rationals also relabeled in the same terms as cuts), Dedekind remarks

> In order to obtain a basis for the orderly arrangement of all *real,* i. e., of all rational and irrational numbers we must investigate the relation between any two cuts (A_1, A_2) and (B_1, B_2) ...

12. We write $\alpha = (A_1, A_2)$, $\beta = (B_1, B_2)$.

    Define

    i) $\alpha = \beta$

    ii) $\alpha > \beta$

    iii) $\alpha + \beta$

    iv) $\alpha \cdot \beta$, and *prove* that $\sqrt{2}\cdot\sqrt{3} = \sqrt{6}$.

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The Irrational Numbers

Answer sheet: Dedekind and the definition of the irrationals.

1-4. Dedekind's definition of the irrationals is based on the rationals and some of their properties. Therefore, we begin by summarising some of the important properties of the latter.

Suppose that a is the rational number closest to 1/2 (either greater or less than 1/2). Then

$$b = \frac{a + \frac{1}{2}}{2}$$ is also rational.

But $$\left|\frac{1}{2} - b\right| = \left|\frac{1}{2} - \frac{a + \frac{1}{2}}{2}\right| = \left|\frac{\frac{1}{2} - a}{2}\right| < \left|\frac{1}{2} - a\right| .$$

Hence b is nearer to 1/2 than a.

It follows that between any two rationals there is an infinity of rationals. But in spite of their "density", there are further "numbers" which are not rationals. For example, in the interval [0.7, 0.71], the "number" $\frac{\sqrt{2}}{2}$ (0.7071...), is not rational.

Similar arguments lead us to the conclusion that there does not exist a number, rational or not, which is closest to $\sqrt{2}$ (see, for example, the Bombelli worksheet).

5. In the set of worksheets on the history of the negative numbers, we brought one way of defining the negative numbers. The rationals can be defined analogously. See also, Richardson, M. *Fundamentals of Mathematics,* Macmillan, 1960, pp. 78-86.

6-7. The purpose of these questions is to express Dedekind's text in modern symbolism, and thus to make sure that the text is understood.

In property III

$$A_1 \cup A_2 = R \text{ (rationals)} \qquad (\text{or } P_1 \cup P_2 = L)$$

$$A_1 \cap A_2 = \emptyset \qquad (\text{or } P_1 \cap P_2 = \emptyset)$$

i) This is false. For if $A_1 = \emptyset$, then $A_2 = R$, and we do not have a separation of R.

ii) False, since $A_1 \cap A_2 = \emptyset$.

iii) True.

iv) False, since every number greater than *a* belongs to A_2.

v) True.

vi)-vii) "the number *a* itself may be assigned at pleasure to the first or second class". Thus *one* of these two statements may be true, but not both.

8. Suppose, on the contrary that A_1 has a greatest number a_1 and that A_2 has a least number a_2. The

$$\frac{a_1 + a_2}{2}$$

is a rational number, such that

$$a_1 < \frac{a_1 + a_2}{2} < a_2$$

and hence belongs to neither A_1 or A_2 - which contradicts the definition of separation.

9. Dedekind states the axiom of continuity as follows:

> **"If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions."**

Let us rewrite it in terms of *the system R* of rational numbers:

"If all *numbers* of the *system R* fall into two classes such that every *number* of the first class *is less than* every *number* of the second class, then there exists only one *number* which produces this division of all *numbers* into two classes, this severing of *the system R* into two portions".

Since "number" here means "rational number", this statement is not only not an axiom, but it is demonstrably false. The following is an example of a separation of R in the sense of Dedekind, which is not defined by any (rational) number.

A_2 is the set of all positive rationals whose square is greater than 2,

and A_1 the remaining rationals.

10. (A_1, A_2) as given at the end of question 9, define $\sqrt{2}$.

11. In the previous question we used an "intuitive" property of $\sqrt{2}$ (i.e. that its "square" is 2) to help us find the appropriate cut, which then *becomes the definition of* $\sqrt{2}$. Logically, of course, we do not know what $\sqrt{2}$ is until it is defined; therefore, the question, as stated is illegitimate.

The same applies to this question. What is π ? Until we have defined it, we do not know what it is. The intention, however, is clear. The question is asking for the determination of a cut which corresponds to our previous intuitive knowledge of π. This is far from easy: any "formula" for π involves consideration of some infinite process, which in turn requires the mathematical concept of convergence. And it was precisely for this reason, among others, that Dedekind felt the need to supply a pure mathematical definition of continuity in arithmetic; that is, to define the real numbers.

Having developed the concept of convergence, one can prove, for example, that the series

$$4(1 - \frac{1}{3} + \frac{1}{7} - \frac{1}{9} + \ldots .)$$

converges. The sum of this series may then be taken to be the mathematical definition of π.

It is then superfluous to find the Dedekind cut which (also) defines π, but this may be done by comparing any rational with the sum of the series, and assigning it to one of two classes according to whether it is greater or less than this sum.

12. For each of the required definitions, it is sufficient to determine the conditions or properties of one class; the second class is then the remainder of the rational numbers. Also (see remarks in answer to question 11), we may clearly choose definitions at will - unless they lead to contradictions. But this is rarely a worthwhile exercise. The implication in the question is that we should be guided by our previous intuitive experience and, in particular, that the definitions, which will also apply to the rationals since they are included in the cuts, should accord with those definitions already given for the same concepts or operations for the rational numbers.

We use the notation $\alpha = (A_1, A_2)$, $\beta = (B_1, B_2)$

i) The "obvious" definition would seem to be

$$\alpha = \beta \iff A_1 = B_1 ,$$

but, unfortunately, this is "wrong". For, if the cut is rational (i.e. defined by a rational number), it could happen that the rational number, in one case, belongs to the first class, and in the second case to the second class, and nevertheless the cuts must be regarded as the same, since they define the same rational number.

Therefore, we define

$$\alpha = \beta \iff A_1 = B_1 ,$$

or if there is one and only one element in A_1 (B_1) which is not in B_1 (A_1) .

ii) If A_1 contains more than one rational number that are not contained in B_1, then we define

$$\alpha > \beta \iff A_1 \supset B_1 .$$

It can be proved that the relation > , as here defined, has all the properties which are already familiar from the set of rationals.

iii) $$\alpha + \beta = \gamma$$

where $\gamma = (C_1 , C_2)$ and is defined by

$$C_1 = \{c_1 \mid a_1 + b_1 = c_1 , a_1 \varepsilon A_1 , b_1 \varepsilon B_1\}$$

iv) To define $\alpha \cdot \beta = \gamma$, we have to distinguish between a number of cases

- if $\alpha > 0$, $\beta > 0$ (according to the definition in section ii), $\alpha > 0$ if A_1 contains rationals greater than zero)

then we define

$$C_2 = \{c_2 | c_2 = a_2 b_2 , a_2 \varepsilon A_2 , b_2 \varepsilon B_2\}$$

- if $\alpha < 0$, $\beta < 0$, then

$$C_2 = \{c_2 | c_2 = a_1 b_1 , a_1 \varepsilon A_1 , b_1 \varepsilon B_1\}$$

- if $\alpha < 0$, $\beta > 0$, then

$$C_1 = \{c_1 | c_1 = a_1 b_2 , a_1 \varepsilon A_1 , b_2 \varepsilon B_2\}$$

The reason for these separate cases, should be clear. If not consider a numerical example of each.

To prove that $\sqrt{2}\cdot\sqrt{3} = \sqrt{6}$, we first note that

$$\sqrt{2} = (A_1 , A_2), \text{ where } A_2 = \left\{x | x > 0 \text{ and } x^2 > 2\right\}$$

$$\sqrt{3} = (B_1 , B_2), \text{ where } B_2 = \left\{y | y > 0 \text{ and } y^2 > 3\right\}$$

By the first case in our definition of the product, we have

$$\sqrt{2}\cdot\sqrt{3} = (C_1 , C_2) , \text{ where } C_2 = \left\{z | z = xy, x \varepsilon A_2, y \varepsilon B_2\right\}$$

Since $x > 0$ and $y > 0$, it follows that $z = xy > 0$

Also $x^2 > 2$ and $y^2 > 3$ $\qquad z^2 = (xy)^2 = x^2y^2 > 6$

Hence the cut (C_1 , C_2) corresponds to $\sqrt{6}$.

Note 1 In this worksheet we have seen how Dedekind "completes" the set R (rationals) to obtain the real numbers. Dedekind, himself, continues by proving that in the extended set we have obtained the desired parallelism between numbers and points:

Every real number produces a division of the set into two classes and every division is formed by some real number. (Remember, it is the second half of this statement which fails in the case of the rationals.)

Note 2 There are alternative definitions of the real numbers. For example, Cantor is credited with the definition using infinite sequences of rational numbers. (See, for example, Goffman, C., *Real Functions,* Rinehart, 1953, p. 28-41.)

THE WEIZMANN INSTITUTE OF SCIENCE
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Equations

A source-work historical collection
for in-service and pre-service
teacher courses

A. Arcavi
M. Bruckheimer

June, 1985

THE WEIZMANN INSTITUTE OF SCIENCE
REHOVOT · ISRAEL
DEPARTMENT OF SCIENCE TEACHING

Equations

Introduction and commentaries to the set of worksheets on the history of linear and quadratic equations

The attached materials are designed for use in workshops for pre and in-service teacher training, in particular for teachers in junior-high schools. The sheets have as their main concern linear and quadratic equations, but touch on other points concerning the mathematics of junior high school and its teaching.

Thus the general aims of this and other sequences of worksheets is:

1- to *improve* the teachers' *mathematical knowledge* of topics included in the curriculum, and that in a way which motivates the teacher to reconsider topics previously studied, but possibly imperfectly understood.
2- to *enrich* the teachers' mathematical background to topics in the curriculum;
3- to allow opportunity for the *discussion of relevant didactics* and to consider them in relation to the pure mathematics of the topics concerned;
4- to create a *reasonable image of mathematics* and mathematical activity as a human endeavour. In particular, to create an awareness of the history of topics included in the curriculum.

In general, the worksheets in this sequence have the following form:

- briefs biographical-chronological introductions in order to set the historical scene,
- an historical source; as far as possible a primary source,
- leading questions on the source material and on mathematical and didactical consequences thereof.

To each worksheet an extensive discussion of the solutions, and points arising from them, is given in the respective *answer sheet*. Thus, the answer sheets contain not only the detailed solutions to the questions, but further source material, background, historical and mathematical information. Both the worksheets and the answer sheets are designed as learning materials.

At the present time, this sequence contains 5 worksheets, which are intended to be worked in the following order:

1. The Rhind Papyrus 2. The Rule of False Position	linear equations
3. Babylonian Mathematics 4. Euclid and the Elements 5. Al-Khowarizmi	on quadratic equations

We have used the sheets in an in-service workshop.
The work is usually guided by a tutor. Participants are given the first worksheet. They work in groups or individually, with the tutor's "interference" if necessary. Then a collective guided discussion takes place, and the answer sheet is distributed. And so on.

For use with Israeli teachers, we translated the sources in the worksheets freely into Hebrew and gave both the original with the translation, in order to encourage the student to read both (if he understood the original language, if not he could look at it and get some flavour of the period by the form of the print, its elaboration, etc., or even try to identify key words etc. from the translation.) Also the original should be available in cases of misunderstanding attributable to mistranslation. In the present English version there are some extracts in languages other than English. Some extracts, for which we did not find an "authorized" English translation, are brought in the original only. For English speaking users of these materials, it may be advisable to translate these latter texts freely into English, alongside the original.

This series is not a text. Therefore there is a need for the tutor using the sheets, to add comments, to be in a position to answer questions, and to have in mind the overall structure of the set.

The stages in the history of equations represented in these worksheets, are far from being complete. There are certainly further sheets which could be added, for example, one on the work of Diophantus and also sheets on more modern times, e.g. Descartes and Frend (see the sequence on negative numbers). Furthermore, this series could be enlarged by adding the interesting history of cubic equations, etc., which although not relevant to the junior-high school curriculum, is relevant to teachers' background. In the following we shall comment briefly on the content of each worksheet.

Commentary on the individual worksheets

1. The Rhind Papyrus

This worksheet presents a little of Egyptian mathematics as it appears in the Papyrus: the number symbolism and some arithmetical operations. The work continues with a guided deciphering of hieroglyphics and some Egyptian arithmetic. This introduction is a preparation for work on the way in which linear equations are solved in the papyrus in comparison to the modern method of solution.

2. The Rule of False Position

This worksheet is a continuation of the previous, presenting the rule of false (single and double) position as arithmetical methods of solving linear equations. Three extracts are brought; from Peletier's Arithmetique (1549), from Tartaglia's General Trattato di Numeri (1556) and from an English arithmetic book (1815). The mathematical justification of the rule in its two versions, its comparison with the algebraic method of solution and some didactical considerations are central issues in this worksheet.

3. Babylonian Mathematics

This worksheet presents a little of cuneiform writing and base 60 arithmetic, in order to discuss the kinds of quadratic equations solved by the Babylonians.

4. Euclid and the Elements

Euclid's famous masterpiece is presented briefly in order to introduce two theorems from Book II. The propositions of this book are considered as "geometrical algebra", although there is no trace of the algebra as we know it today.

The main issue is to see how a geometrical problem can be algebraically interpreted in different ways, and subsequently how the solution of a quadratic equation can be regarded geometrically.

5. Al-Khowarizm

This worksheet brings some other ways of solving quadratic equations, again with interesting geometrical illustrations.

Final Comment

We do not see these sheets as final or definitive. As we find new and or "better" sources, or in the light of experience in the use of the sheets, we make alterations, corrections, replacements, additions, etc.

We would welcome comments from anyone reading these worksheets.

A. Arcavi
M. Bruckheimer

JUNE 1985.

THE WEIZMANN INSTITUTE OF SCIENCE
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Equations

Worksheet : The Rhind Papyrus

The Rhind papyrus is one of the oldest extant mathematical documents. The papyrus got its name from Henry Rhind, an Englishman who bought it in Luxor, Egypt in 1858, and after his death it came into the possesion of the British Museum. The papyrus is also associated with the name of Ahmes, the Egyptian scribe who copied it. It is estimated that the papyrus dates from the 17th century BCE, but is, apparently, a copy of even earlier sources.

The papyrus contains a collection of 87 problems and their solutions. The problems are taken from various topics - arithmetic, calculation of areas, etc.

In this worksheet, we shall look at various extracts dealing with arithmetical operations and what we would call today, the solution of equations of one variable.

I. Hieroglyphics, Hieratics and Egyptian arithmetic

Ancient Egyptian writing* had two forms, hieroglyphic and hieratic. Hieroglyphics are mainly to be found as inscriptions on stone in temples and sepulchres. Hieratic is a cursive script, quicker to write, used mainly in papyri.

Below we bring some of the hieroglyphic symbols, which are easier to decipher than in hieratic script, which differed from scribe to scribe.

The hieroglyphic number symbols were:

1 =	I	1,000 =		(lotus flower)
10 =	∩	10,000 =		(bent finger)
100 =	9	100,000 =		(tadpole)
		1,000,000 =		(man with raised arms)

The Egyptian used only fractions with numerator 1 (except for 2/3). These fractions are called unit fractions: i.e. of the form $\frac{1}{n}$, $n \in N$. Other fractions were written as the sum of unit fractions.**

They write fractions as follows:

$\frac{1}{10}$ = 

$\frac{1}{8}$ = 

The symbol (an open mouth) played the role (in our terms) of the fraction bar with numerator 1

$\frac{2}{3}$ = 

$\frac{1}{2}$ =

} special symbols

---

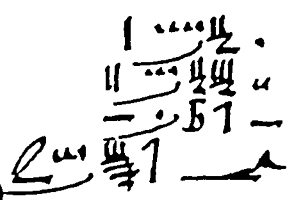
* Later scripts (from the 10th century BCE onwards) are the Demotic and Coptic, which we do not use in this sheet.

** Unit fractions are described more fully in, for example, Beck, A., Bleicher, M.N. & Crowe, D.W., Excursions into Mathematics, Worth Pub., 1969.

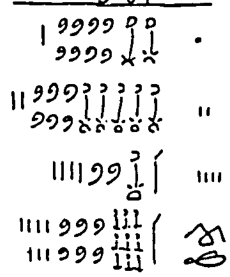
The Rhind papyrus is written in Hieratic script.
The following are two examples of calculations performed as parts of the solution of problems in the papyrus.
We give them in hieratic, hieroglyphic and modern notation*.

From the solution of Problem 74.

Hieratic



Hieroglyphic

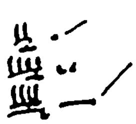


Modern

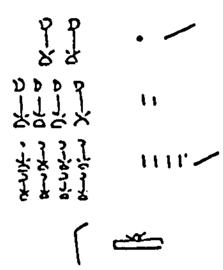
1	2801
2	----
4	----
Total	19607

From the solution of Problem 52.

Hieratic



Hieroglyphic



Modern

1	----
2	----
4	----
Total	10000

---

* The passages are taken from
Chace, A.B., The Rhind Mathematical Papyrus, NCTM, 1979.
Peet, T.E., The Rhind Mathematical Papyrus, University of Liverpool Press, 1970.

Questions

1. Note three characteristics of the Egyptian method of writing numbers in hieroglyphics.

2. Complete the blanks in the "Modern" column.

3. a) Explain the calculation and its method in each of the two cases.
   b) What is the difference between the two cases? (Note the mark / against certain numbers.)

4. Calculate 13 x 27 by the Egyptian method.

5. Can one multiply any pair of numbers by the Egyptian method? Explain.

II. The solution of equations in the Rhind papyrus

Problems 24-38* fall into a class by themselves, having in common the fact that their solution involves solving a linear equation in one variable.

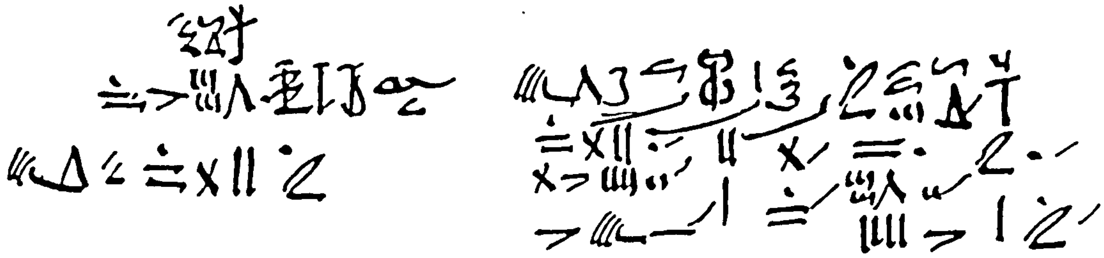
Some of these problems are termed problems of *'aha'*, since this word appears in them, and it would seem that its use is similar to our use of "variable".

The following is Problem 24 (the English translation being taken from the book by Peet)

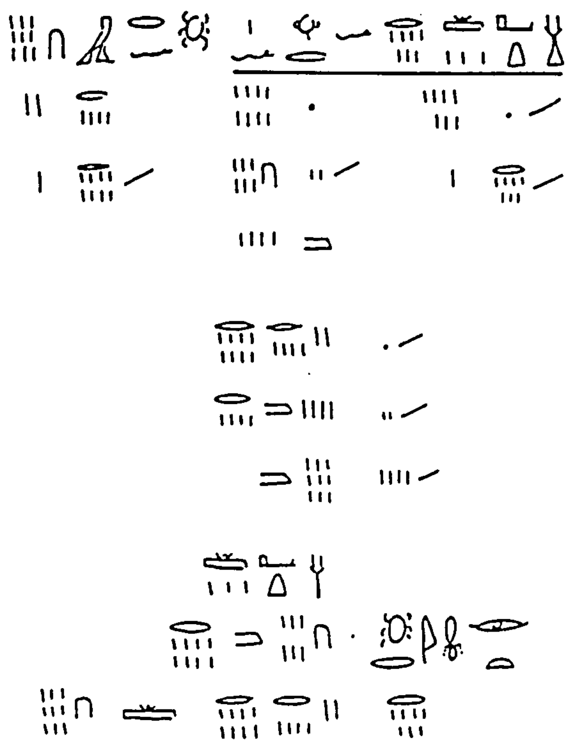
---

* In the original, the problems are not numbered; this was done by the commentators.

Hieratic*



Hieroglyphic**



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* Chace, page 97.

** Peet, appendix.

Translation*

A quantity whose seventh part
is added to it becomes 19.

⁄ 1	7
⁄ $\frac{1}{7}$	---
1	8
⁄ 2	---
$\frac{1}{2}$	---
⁄ $\frac{1}{4}$	---
⁄ $\frac{1}{8}$	---
⁄ 1	$2 + \frac{1}{4} + \frac{1}{8}$
⁄ 2	-----
⁄ 4	-----

The doing as
it occurs :—The quantity is $16\frac{1}{2} + \frac{1}{8}$
one seventh is ---
Total ---

* As it appears in Peet. The ommisions are ours (see the following questions.

Questions

6. Write, in modern notation, the equation corresponding to the first sentence, and solve it.

In the following questions we shall decipher the method used by the Egyptians.

7. Look at the first step:

/	1	7
/	$\frac{1}{7}$	----

Apparently the Egyptians approached the problem by substituting a trial number, and saying what happens.

a) Which number did they try ?

b) Complete the blank.

c) What result was obtained ?

d) Why do you think they chose the number they did ?

8. Look at the second step:

	1	8
/	2	----
	$\frac{1}{2}$	----
/	$\frac{1}{4}$	----
/	$\frac{1}{8}$	----

a) Complete the blanks.

b) Which calculation has been performed, and what is the result ?

9. Look at the third stage:

/ 1	$2 + \frac{1}{4} + \frac{1}{8}$
/ 2	---------
/ 4	---------

a) Complete the blanks.

b) Which calculation has been performed, and what is the result ?

10. Look at the final stage:

The doing as
it occurs :—The quantity is $16\frac{1}{2} + \frac{1}{8}$
one seventh is ____
Total ____

Complete the blanks and explain what has been achieved in this step.

11. Reproduce and summarize the method of solution and explain it.

12. Write down the solution of the problem, as it would have appeared in the papyrus (but in modern notation), if the first trial number had been 14 instead of 7.

13. Problem 25 in the papyrus is

A quantity whose half is added to it

becomes 16.

Solve the problem as you would today and using the Egyptian method (as you think it would appear in the papyrus).

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1985

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DEPARTMENT OF SCIENCE TEACHING

Equations

Answer Sheet: The Rhind Papyrus

1. Characteristics of hieroglyphic numeration include

   a) The writing is from right to left.

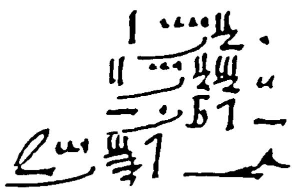
   b) The system is essentially decimal; that is one symbol which represents ten identical "lesser" symbols.

   c) Numbers are formed by the juxtaposition of symbols, but there is no place value; that is, if the symbols designating a number are rearranged, they still represent the same number.

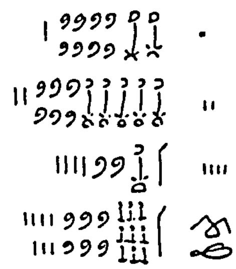
   d) As it may follow from c) there is no symbol for zero. The number 201, for example, is written ı99

2. From Problem 74.

Hieratic



Hieroglyphic

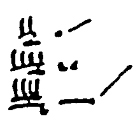


Modern

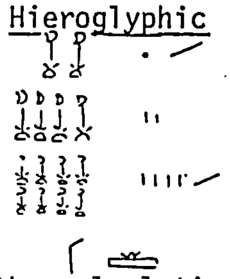
1	2801
2	5602
4	11204
Total	19604

From Problem 52.

Hieratic



Hieroglyphic



Modern

/	1	2000
	2	4000
/	4	8000
Total		10000

3. In the first case the calculation is :

$$1 \times 2801 = 2801$$
$$2 \times 2801 = 5602$$
$$4 \times 2801 = 11204$$

Total $1 \times 2801 + 2 \times 2801 + 4 \times 2801 = 19607$

i.e. $(1 + 2 + 4) \times 2801 = 19607$

$$7 \times 2801 = 19607$$

In other words, the calculation is multiplication by adding multiples of the number by 2.
The same method is used in the second case to calculate 5 X 2000, but here we do not take all the multiples of 2000, only those indicated by / .

4.

/	1	27	or	/	1	13
	2	54		/	2	26
/	4	108			4	52
/	8	216		/	8	104
				/	16	208
Total		351	Total			351

5. This question is answered by noting that any number can be written as the sum of powers of 2, as we do in binary arithmetic.

6. The given sentence is equivalent to the equation. Solution:

$$x + \frac{1}{7}x = 19$$

$$\frac{8}{7}x = 19$$

$$x = \frac{19 \cdot 7}{8} = 16\frac{5}{8}$$

7. In the papyrus, the first step is :

/	1	7
/	$\frac{1}{7}$	1

That is, one effectively substitutes 7 in the left-hand side of the equation, and calculates the result, which is 8. It would seem likely that 7 is chosen, because it is then easy to calculate a seventh.

8. The second stage is :

	1	8
/	2	16
	$\frac{1}{2}$	4
/	$\frac{1}{4}$	2
/	$\frac{1}{8}$	1

Here we are apparently trying to find multiples of 8, whose sum will give us 19. That is, we are trying to solve

$$8 \cdot \boxed{?} = 19.$$

The result obtained is $2 + \frac{1}{4} + \frac{1}{8}$, i.e. $2\frac{3}{8}$

9. The third stage is:

/ 1	$2\frac{1}{4}[+]\frac{1}{8}$
/ 2	$4\frac{1}{2}[+]\frac{1}{4}$
/ 4	$9\frac{1}{2}$

At this stage we are calculating the product:

$$(2 + \frac{1}{4} + \frac{1}{8})\cdot(1 + 2 + 4)$$

I.e. $2\frac{3}{8}\cdot 7$

and the result is $16\frac{5}{8}$, calculated as the sum of the three numbers in the right-hand column above.
$16\frac{5}{8}$ is the required number (solution of the equation).

10. The final stage is:

"The doing as it occurs:-The quantity is $16\frac{1}{2} + \frac{1}{8}$
one seventh is $2\frac{1}{4} + \frac{1}{8}$
Total 19 "

This stage is a check on the number obtained

$$\underbrace{16\frac{5}{8}}_{\text{the quantity}} + \underbrace{2\frac{3}{8}}_{\text{a seventh of it}} = 19$$

11. The method is

- "Substitute" 7, and see what we obtain; we obtain 8.
- The number of times we have to multiply 8 (the result of the trial substitution) to obtain 19 (the correct result), by that number we have to multiply 7 (the trial substitution) to obtain the required number.

The justification is that since

$$(1\tfrac{1}{7})\cdot 7 = 8$$

then if $k\cdot 8 = 19$, we have $(1\frac{1}{7})\cdot 7\cdot k = 19$, so that $k\cdot 7$ is the required quantity.

12. If we choose 14 for the trial substitution, the Egyptian solution would appear as follows:

/	1	14
/	$\frac{1}{7}$	2

/	1	16
	$\frac{1}{2}$	8
	$\frac{1}{4}$	4
/	$\frac{1}{8}$	2
/	$\frac{1}{16}$	1

	1	$1\frac{1}{8}[+]\frac{1}{16}$
/	2	$2\frac{1}{4}[+]\frac{1}{8}$
/	4	$4\frac{1}{2}[+]\frac{1}{4}$
/	8	$9\frac{1}{2}$

which, of course, again gives us $16\frac{5}{8}$.

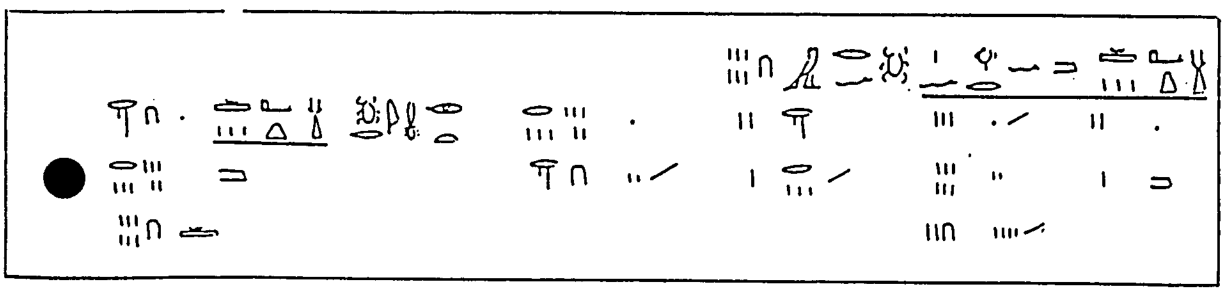
13. $x + \frac{1}{2}x = 16$

$$\frac{3}{2}x = 16$$

$$x = \frac{16 \cdot 2}{3}$$

$$x = 10\frac{2}{3}$$

The following is the solution in hieroglyphics (as given by Peet).



Peet's English translation

A quantity whose half is added to it becomes 16.

1	2
$\frac{1}{2}$	1
/ 1	3
2	6
/ 4	12
$\frac{2}{3}$	2
/ $\frac{1}{3}$	1
1	$5\frac{1}{3}$
/ 2	$10\frac{2}{3}$

The doing as it occurs

:—The quantity is $10\frac{2}{3}$

a half is $5\frac{1}{3}$

Total 16 "

Peet, in his book on the papyrus (page 60), relates the arguments among nineteenth century historians. There were those who credited the Egyptians with knowledge of algebra, and others who objected strongly to this view.

Among the parts in the discussion were the absence of algebraic notation among the Egyptians, and the method of solution.

Even though we do not get involved with this sort of historical discussion, we should be aware of the existence of different and conflicting interpretations.

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1985

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Equations

Worksheet: The Rule of False Position

Linear equations in one unknown were solved by what was essentially the Egyptian method for a very long time. In the Middle Ages this method was called the *Rule of False Position*, or the Rule of False, or the Rule of False Supposition, etc.

In this worksheet we shall have a look at the method as it appeared in mathematics texts at various times and, in the process, we shall learn a little more about it.

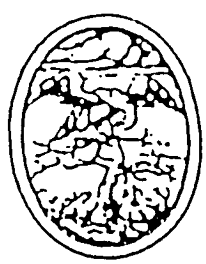
J. Peletier

Jacques Peletier (1517-1582) was a Frenchman writer and doctor, who also engaged in mathematics. The following extract is taken from one of his books, whose frontispiece is shown in the next page.

L'Aritmetique de

IACQUES PELETIER du Mans, departie en quatre Liures,

A THEODORE DEBEZE.



Auec Priuilege du Roi.

On les vend a Poitiers a l'enseigne du Pelican.

M. D. XLIX.

LA Regle de Faux, que les Arabes appellent la Regle Catain, est ainsi ditte, par ce que d'vn cas faux presupposé, elle enseigne a trouuer le vrai. ...

La Regle de Faux d'vne Position a presque pareille operation a celle de la Regle de Trois, excetté qu'en la regle de Trois nous auons trois termes congnuz: ici nous n'en auons qu'vn ... a la semblance duquel nous en formons deux autres, ...

Exemple. I'ai mis certaine somme d'ecuz en Bãque pour en auoir par chacun an 6 pour 100: Au bout de 10 ans m'ont eté baillez 500 Ecuz pour tout. Qu'elle etoit la somme principalle.

... Feignons vn Nombre a plaisir, & par icellui faisons notre discours, tout ainsi que si c'etoit la somme principalle que nous cherchons. Comme par exemple, mettons cas que ce soint 200 Ecuz que i'auoie premierement baillez: donq' ilz m'ont valu en 10 ans 120 Ecuz a raison de 6 pour 100: Or 120 ioint z auec 200 ne font que 320 Ecuz: Mais il en falloit 500. Voilà comment i'ai trois termes pour la regle de Trois: l'vn qui contiendra la question, qui est 500, & les deux autres que i'ai formez artificiellement, qui sont 200 & 320: de sorte que 320 doit auoir telle Proportion a 200, comme 500 a au terme que ie cherche, sauoir est a la vraie somme principalle. I'ai donq' recours a la Regle de 3 en cette sorte ...

Questions

1. Towards the beginning of the extract "la Regle de Trois" (the Rule of Three) is mentioned, and Peletier writes

   "en la regle de Trois nous avons trois termes congnuz"..
   What is the purpose of this rule and how does it work given three values a, b, c?

2. Rewrite the problem in the extract using modern terms.

3. a) Identify the known term ("l'un qui contiendra la question") and the other two which are formed from it ("les deux autres que j'ai formez artificiellement"). Complete the bit missing at the end of the extract, to obtain the solution to the problem.

   b) Divide the solution into steps, and explain what is done in each step.

4. Describe the resulting solution if Peletier had begun with 350 as the initial value, instead of 200.

5. Can we assume any number at the beginning? Explain.

6. Solve the problem algebraically.

In the following we bring two extracts. The first is taken from the booklet *Guess an answer,* a school text for Grade 9, C stream (weakest), as published by the Science Teaching Department of the Weizmann Institute. The second extract is taken from the *Teacher Guide* to the previous, and describes how one might present a lesson on the solution of such equations.

<u>Equation</u> Worksheet A/1

a) Consider the equation $3x = 48$.
Which number shou d one substitute for the variable in the equation to obtain a true statement? _________

(That is, which number when multiplied by 3 gives 48?)

b) ...

c) What must we substitute for the variable in the following equation to obtain a true statement.

i) $x + 2x + 3x = 72$ ____________

ii) $(x + 2)3 = 21$ ____________

A suggested lesson

We begin with Worksheet A/1, and ask the students to guess the answers.

The lesson is organized as individual work, and each student should try to solve the equation in his/her own way. As they work, the teacher talks with them, and discusses various ways of looking for the solution.

a) Substitution of various numbers in place of the variable and checking whether we get a true statement.
b) Searching for the solution by asking suitable questions.
   In question c, ii), for instance:
   "Which number when multiplied by 3 gives 21?"
   If we get the answer 7, then let's agree that the number 7 is "hiding" in the brackets.

   "And now which number do we have to add to 2 to get 7?". The number obtained can be substituted in the original equation to see if we get a true statement.

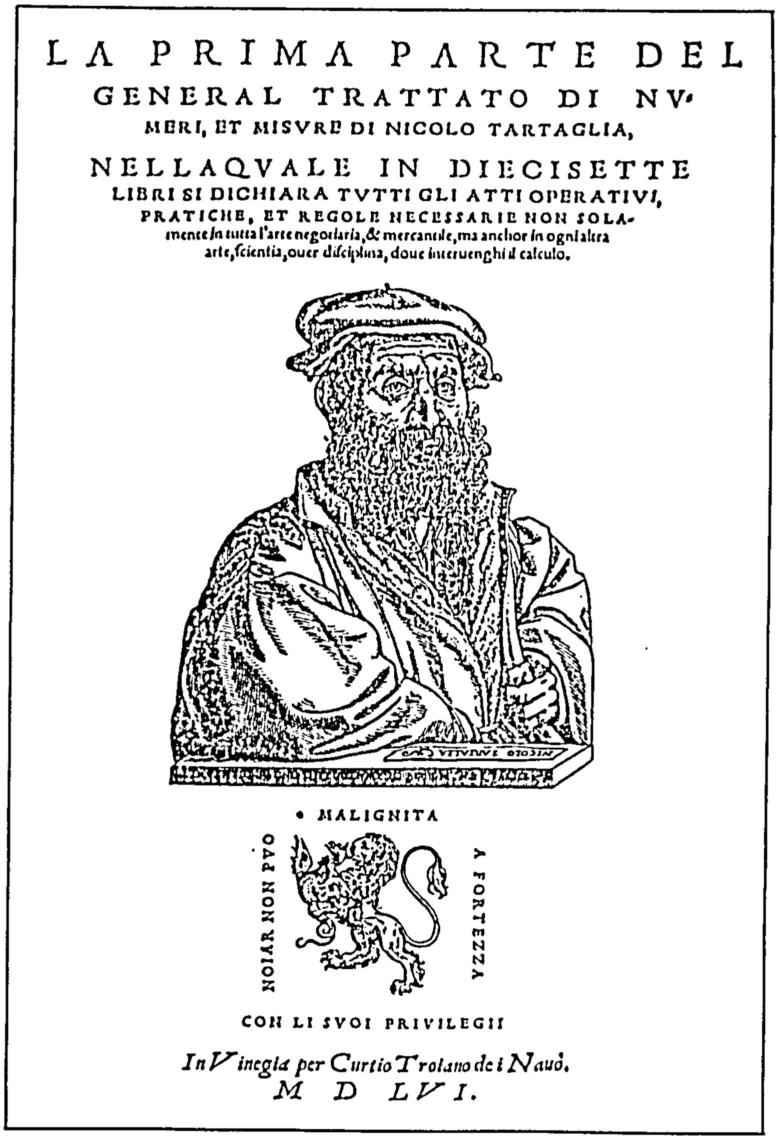
Question

7. What does the method suggested here have in common with, and in what way is it different from, that given by Peletier to solve a linear equation in one unknown?

N. Tartaglia

Nicolo Tartaglia (1506-1557) was one of the famous Italian mathematicians of the 16th century. The story goes that when he was a child, he received facial injuries which caused him to stammer - hence the name Tartaglia - the stammerer.

Tartaglia is credited with solving certain cubic equations.


LA PRIMA PARTE DEL
GENERAL TRATTATO DI NV-
MERI, ET MISVRE DI NICOLO TARTAGLIA,
NELLAQVALE IN DIECISETTE
LIBRI SI DICHIARA TVTTI GLI ATTI OPERATIVI,
PRATICHE, ET REGOLE NECESSARIE NON SOLA-
mente in tutta l'arte negotiaria, & mercantile, ma anchor in ogni altra
arte, scientia, ouer disciplina, doue interuenghi il calculo.

• MALIGNITA
NON PVO
FORTEZZA
CON LI SVOI PRIVILEGII

In Vinegia per Curtio Troiano dei Naud.
M D LVI.

The following extract is taken from the book of Tartaglia, whose frontispiece is depicted on the facing page.

TRE Compagni vogliono far vna compagnia (a far lauorar di lana)& fanno conto, che a principiar tal mercantia non gli vuol manco di ducati 1000. Il ſecondo di detti compagni ſi offerſe di mettere il doppio di quello, che mettera il primo,& il terzo ſi offerſe di mettere il treppio di quello, che mettera il ſecondo. Si adimanda volendo che la ſumma di cio che metteranno fra tutti tre ſia li detti ducati 1000. quanto douera mettere ciaſcun di loro in detta compagnia.

Farai in queſto modo poni che'l primo debba mettere quello che ti pare(perche'l non importa a poner poco,ouer aſſai)hor poniamo che debba mettere ducati 100. ...

Se tutte queſte tre partite ſummate inſieme faceſſero preciſamente li detti ducati 1000. il caſo ſaria riſolto, il che potria alle volte auenir per ſorte

Questions

8. Complete the solution using the same steps which you found in your solution of Question 3.b).

9. Solve the problem algebraically.

10. As a result of your experience with the two problems so far presented, try to generalise: for which type of problem can the solution be found by the *simple rule of false position?*

The rule of double false

The three problems we have seen so far could be solved by the *simple rule of false position,* in which we initially substitute a single arbitrary value.
In the following, we shall have a look at *the rule of double false,* in which we start with two arbitrary values.

The following extract is taken from the 6th edition of the textbook.

Keith Thomas, The Complete Practical Arithmetician: containing Several New and Useful Improvements Adopted to the use of Schools and Private Tuition,
published in London in 1815.
As usual, we have omitted bits for the purpose of the exercise.

DOUBLE POSITION.

. . .

RULE.

Suppose any two convenient numbers, and proceed with them according to the nature of the question, marking the errors (with + or —) according as they exceed or fall short of the truth.

Then,

Multiply the first supposition by the second error, and the second supposition by the first error, and divide the sum of the products by the sum of the errors, if they are differently marked; or the difference of the products by the difference of the errors, if they are marked alike, and the quotient will be the number sought.

. . .

Examples.

(1.) What number is that, which, being multiplied by 3, the product increased by 4, and that sum divided by 8, the quotient may be 32?

Suppose 12	Again, suppose 108
×....	×....
....	....
+....	+....
8)	8) ....
Quotient	Quotient ....
should be 32	should be 32
Error — 27	Error + 9

By Rule I.

			its error.
1st supposition	12		—27
2d supposition	108		+ 9
	27	its error	12
	....		....
	....		

27+9=36)3024(84 answer.

Questions

11. Explain the author's meaning, when he writes

    - "two convenient numbers"
    - "proceed with them according to the nature of the question"
    - "error".

12. Complete the blanks.

13. Use the rule of double false, with the values 90 and 105, in the problem in the extract from Tartaglia.

14. Use the simple rule of false position, with the value 52, on the present problem.

15. From your answers to Questions 13 and 14, conclude when it is legitimate to use the simple, and when the double, method of false position.

16. Justify algebraically the rule of double false position.

The following extracts deal with the value of *the rule of false position*. The first two extracts are from the sources we have already used.

From Peletier - 1549

> ... *Et eſt celle de toutes les Regles vulgaires, de laquelle l'vſage eſt plus beau & plus ample.*

From Tartaglia - 1556

DVE altre ſpecie di Regole, nella Pratica di Numeri, ...
... ,l'una dellequali è detta Poſition Sempia, & l'altra è chiamata Poſition Doppia, con lequali quaſi tutte le queſtioni, che per le altre regole per auanti date ſono ſtate riſolte, ſi riſoluerebbono, & oltra di queſto infinite altre, ...
ſe ne riſolueranno, che per niun'altra regola (dico di quelle per fin al preſente date) ſaria poſſibile di poter riſoluere.

From Hatton* - 1731

... I ſhall not inſiſt farther, becauſe the Solution of an eaſy ſimple Equation in Algebra anſwers not only any Queſtion in this Rule, but alſo gives at the ſame time a Canon whereby any Queſtion of the like kind is much more eaſily and ſpeedily reſolved.

Questions

17. Summarise the various approaches to *the rule of false position,* and explain the differences between them.

18. List the mathematical operations required in each of the methods *rule of false position* - modern algebraic method.
Compare them with regard to didactic difficulty.

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* Hatton, E., An Intire System of Arithmetic. 2nd. edition, London, 1731.

A. Arcavi
M. Bruckheimer

THE WEIZMANN INSTITUTE OF SCIENCE
REHOVOT · ISRAEL
DEPARTMENT OF SCIENCE TEACHING

Equations

Answer sheet: The rule of false position.

1. The purpose of "the rule of three" is to find a value (number) which is related in proportion with three given values (numbers). In algebraic form, given a, b, c such that

$$\frac{a}{b} = \frac{x}{c} \quad \text{or} \quad \frac{a}{b} = \frac{c}{x},$$

then $$x = \frac{ac}{b} \quad \text{or} \quad x = \frac{cb}{a}.$$

Nowadays the rule of three has almost completely disappeared from maths text books. But, the long history of the rule, in all its forms, is very interesting.

The rule of three is clearly connected with the concepts of ratio and proportion. The mathematical historian, D.E. Smith*, claims that the simple aspects of these concepts are already in evidence in the earliest calculations done by man. In ancient Greece, these concepts occurred in geometry. Thus, in Euclid's *Elements***,

---

* Smith, D.E., *History of Mathematics*, Vol. II, Dover, 1958, p. 477 ff.

** Heath, T.L., *The Thirteen Books of Euclid's Elements*, Dover, 1956.

we found the rule of three in its geometric version. It takes the form of finding a proportional line segment, given three other segments.

Nevertheless, the rule of three is connected with ratio and proportion, as a method of finding the solution to certain problems (in particular, commercial problems). It appears, apparently for the first time under this name, in the work of the Hindu mathematician Brahmegupta*, in the 7th century. The rule of three appeared in most arithmetic texts of the Middle Ages, as well as in more modern times.

2. A certain sum of money is deposited in the bank for 10 years, with annual interest of 6%. If at the end of the period a sum of 500 ducats is realised, what was the original sum?

3. a) The known term ("l'un qui contiendra la question") is 500, and those formed are 200, the guess (mettons cas que ce soit ...") and 320, the result of the substitution according to the conditions stated in the problem. Using these three values, the solution can be completed by the rule of three, as follows

   Si 320 Ecuz prouienent de 200, de combien prouienent 500? Multipliez 500 par 200, ce sont 100000, lesquelz diuisez par 320 sont 312 $\frac{1}{2}$, qui est la somme que i'auoie baillee.

---

* Some of his work was translated into English in Colebrooke, H.T., *Algebra with Arithmetic and Mensuration, from the Sanscript of Brahmegupta and Bhascara,* London, 1817.

b)

Step	General description	In the example of the extract
Step 1	Choose any number	200
Step 2	Apply the conditions of the problem to the chosen number	6% (a year) of 200 is 12. After 10 years, 120*. Hence, the capital plus interest (after 10 years) is 320.
Step 3	Apply the rule of three to the three numbers	320 ⟶ 200 500 ⟶ x Hence $x = \frac{500 \times 200}{320}$ $= 312\frac{1}{2}$

These steps characterize the simple rule of false position.

4. Step 1 : Choose 350.

Step 2 : 6% (annually) of 350 is 21.
After 10 years the interest is 210.
Capital plus interest after 10 years is 560.

Step 3 :
560 ⟶ 350
500 ⟶ x

Hence $x = \frac{500 \times 350}{560}$

$= 312\frac{1}{2}$

Here we chose a different number, greater than the actual solution, and still obtained the correct answer.

---

* See note in page 4 .

5. The previous question suggests that we can start with any number. We shall use algebra (which was not yet in use in Peletier's time), to show that the result is independent of the chosen number.

If we chose to start with the number a, then after 10 years the total interest will be 0.6a, giving a total of 1.6a. Then by the rule of three,

if 1.6a results from a
then 500 results from x

where $$x = \frac{a \times 500}{1.6a} = \frac{500}{1.6} = 312.5.$$

6. $a + 10 \times 0.06a = 500$

$$1.6a = 500 , \quad a = 312.5.$$

Note: This is the algebraic solution of the problem in the extract, from which it follows that simple interest is intended. If the interest had been compound, then the corresponding algebraic form would have been

$$1.06^{10} \cdot x = 500 ,$$

whose solution is approximately 279.20.

7. Peletier's method and that in the booklet *Guess an answer* have the first step in common - the guess - which is then substituted and one calculates the outcome according to the conditions of the problem. The following extract from the *Teacher Guide*, emphasises the importance of the guess as a first step in the sclution:

This is the student's first meeting with equations, and at this stage we suggest that the solution be found by guessings for the following reasons.

The student does not yet have any techniques for solving equations. Guessing replaces technical manipulations.
Everyone, whatever level he is at, sometimes uses guessing. Guessing is sometimes quicker than a standard technique. Guessing gives a clearer feeling for the size of number and its significance. Guessing gives a further opportunity to practice arithmetical calculations.

The difference between the two methods is summarized as follows.

Peletier	Guess an answer
The problem is expressed in words.	The problem is an algebraic exercise.
The guess is used to "create" two further values, in order to use the rule of three - i.e. an algorithm for finding the answer.	The guess is used to give a feel for the size of the number being sought. There is no algorithm.
The solution is purely arithmetic - no algebra is involved	The answer is obtained by leading questions, which are the verbal form of the algebraic techniques one wishes the student to learn later.

A third method, which is a sort of compromise between the other two, can be used, whether the problem is given in verbal or algebraic form. (We are not concerned, here, with the possibility of "translation" from words to algebra.) Thus, we again begin with a guess, substitute in the conditions of the problem, and get a feel for the size of the answer we are looking for. The second time round, we again use a guess, but with the intention of "correcting" intuitively the error resulting from the first guess. Then, we consider the results and continue the process of guessing according as the results are "too big" or "too small". It is possible that this method will involve, in some cases, a lot of work, if we want an exact answer, but we need no techniques - neither arithmetic (the rule of three) nor algebraic. The process depends on the feel for estimation and approximation, which is a valuable asset, and can improve with use.

8. <u>First step</u>: guessing.

hor poniamo che debba mettere ducati 100. ...

<u>Second step</u>: using the conditions of the problem.

... Ma perche in questo nostro caso summando insieme le dette tre partite, cioe li ducati 100.& li ducati 200.& li ducati 600 fanno solamente ducati 900.& non ducati 1000. e pero la nostra positione è stata falsa, nondimeno con tal falsita potremmo ritrouar la verita.

Third step: the rule of three.

Dicendo per la regola del 3. Se ducati 900. vien da ducati 100. & da ducati 200. & da ducati 600. da chi venira ducati 1000. opera come vuol la regola, & trouarai che'l primo douera mettere ducati $111\frac{1}{9}$, il secondo ducati $222\frac{2}{9}$, il terzo ducati $666\frac{6}{9}$ ...

Note Tartaglia states explicitly (but without proof) that we may choose any number, but he may well be suggesting that we estimate the answer, when he writes that it may happen that we solve the problem with the first guess. But, in most problems, the numbers are such that the first guess will solve the problem by chance only.

9. $$\underbrace{x}_{\text{Investment of first partner}} + \underbrace{2x}_{\text{Investment of second partner}} + \underbrace{3\cdot 2x}_{\text{Investment of third partner}} = 1000$$

I.e. $9x = 1000$

whence $x = 111\frac{1}{9}$

10. As we have seen, the *simple rule of false position* is based on the rule of three, and the latter is designed to find the "fourth proportional". In the two problems we have met so far, the elements are in direct proportion. The simplified algebraic forms of the two problems are

Problem I $1.6x = 500$

Problem II $9x = 1000.$

And we conclude that, in general, the rule of false position can be used in problems whose simplified form is $ax = b$.

The general solution, then becomes

a) Chose a number, m say.

b) Substitute, to obtain am = p, say.

c) Use the rule of three

$$m \longrightarrow p \qquad \text{or } m \div p = x \div b$$
$$x \longrightarrow b$$

so that $x = \frac{mb}{p}$.

11. - "two convenient numbers" are numbers which will prove convenient in the calculations determined by the conditions in the problem.

- "proceed with them according to the nature of the question" means substituting the two chosen numbers in the conditions of the question and doing the resulting calculations.

- "error" is the difference between the value obtained (as a result of the guess) and the true outcome. For example, in the second problem above, the error is 900 - 1000 = -100.

12. The solution, as given in the original, is as follows.

Suppose 12	Again, suppose 108
×3	×3
36	324
+4	+4
8)40	8)328
Quotient 5	Quotient 41
should be 32	should be 32
Error —27	Error +9

By Rule I.

			its error.
1st supposition	12	X	—27
2d supposition	108		+ 9
	27	its error	12
	2916		108
	108		

27+9=36)3024(84 answer.

The use of the X symbol in the double rule of false, already occurs in texts in the 13th century. Subsequently it was used widely until the 19th century.

13.

	First supposition	Second supposition
First partner	90	105
Second partner	180	210
Third partner	540	630
	810	945
	1000	1000
First error	190 (negative)	Second error 55 (negative)

1st supposition	90	X	190 error (negative)
2nd supposition	105		55 error (negative)
	19950		4950

Since the errors are of the same sign, one has to divide the difference of the products by the difference of the errors.

$$\frac{19950 - 4950}{190 - 55} = \frac{15000}{135} = 111\frac{1}{9}$$

14. Starting with 52, we obtain

$$52 \cdot 3 = 156$$

$$156 + 4 = 160$$

$$160 : 8 = 20$$

and by the rule of three

$$20 \longrightarrow 52$$

$$32 \longrightarrow x$$

Hence $$x = \frac{32 \cdot 52}{20} = 83\frac{1}{2}$$

In this case, the simple rule of false does not give us a correct solution.

If we look at what we have done in general form; i.e., suppose we start with the number a

$$a \longrightarrow \frac{a \cdot 3 + 4}{8}$$

$$32 \longrightarrow x$$

Hence $$x = \frac{\frac{a \cdot 3 + 4}{8} \cdot 32}{a} = \frac{12a + 16}{a}$$

In contrast to what we say in Question 5, the result here depends on the value of a, and hence the method does not work.

15. Keith gives the conditions for using the simple or double rule of false as follows.

SINGLE POSITION.

Definition. *By Single Position*, or a single supposition, are solved those questions wherein the results are proportional to their suppositions.

DOUBLE POSITION.

Definition. *By Double Position*, or two suppositions, are solved those questions wherein the errors are proportional to the difference between the true number, and each supposition.

If we write the types of question in algebraic form, then

i) a problem whose unsimplified algebraic form is

$$ax + bx + cx + \ldots. + mx = p$$

can be solved by the simple rule.

ii) a problem whose unsimplified algebraic form is

$$ax + bx + cx + \ldots. + mx + k = p$$

can be solved by the double rule.

We can also conclude that problems of type i) can also be solved by the double rule of false (prove it!), but type ii) cannot be solved by the simple rule.

16. If we substitute p_1 and p_2 in $ax + b = 0$, we obtain

(1) $ap_1 + b = s_1$

(2) $ap_2 + b = s_2$

In this (general) case, s_1 and s_2 are both the results of the substitution as well as the error, since the right-hand side of the given equation is zero.

Substracting (2) from (1) we obtain

$$a(p_1 - p_2) = s_1 - s_2$$

whence $$a = \frac{s_1 - s_2}{p_1 - p_2}$$

Multiply 1) by p_2 and (2) by p_1,

$$ap_1p_2 + bp_2 = s_1p_2$$

$$ap_2p_1 + bp_1 = s_2p_1$$

Subtract $$b(p_2 - p_1) = s_1p_2 - s_2p_1$$

whence $$b = \frac{s_1p_2 - s_2p_1}{p_2 - p_1}$$

but since $x = -\frac{b}{a}$, $$x = \frac{\dfrac{s_1p_2 - s_2p_1}{p_2 - p_1}}{\dfrac{s_1 - s_2}{p_1 - p_2}} = \frac{s_1p_2 - s_2p_1}{s_1 - s_2}$$

and this is the *double rule of false.* If we substitute for s_1 and s_2 the errors together with their appropriate signs, then the formula includes both possible cases of error - which exemplifies the way algebra makes it easier to generalise.

17. The first two quotations praise *the rule of false position* as a method of great power in the solution of many problems. The third, on the other hand, mentions the method, but claims that, using algebra, we can solve the same problems, more quickly, more easily and more generally.

The reason for the difference in an approach can be found in the historical development. In the middle of the 16th century, there was, as yet, no algebraic symbolism, whereas in the 18th century, it was already very similar to what it is today. As a result of this development, *the rule of false* appeared less and less - even though there are still 19th century texts which bring the rule without explanation. Today it is almost unknown, except for books on the history of mathematics, but the tenacity of this and other special methods in the face of the algebraic power of generality can be judged from the following, written by John Perry, a maths education "revolutionary", in 1899 (in the book whose title page is reproduced on the next page).

19. Those parts of arithmetic called "Equation of Payments," "Barter," "Profit and Loss," "Fellowship," "Alligation" of many kinds, "Position," "Double Position," "Conjoined Proportion," and many others, are, when we strip them of their technical terms and artificial complexities, the simplest of algebraic exercises, and they ought to be treated as such.

DEPARTMENT OF SCIENCE AND ART OF
THE COMMITTEE OF COUNCIL ON EDUCATION,
LONDON.

PRACTICAL MATHEMATICS.

SUMMARY OF SIX LECTURES DELIVERED TO
WORKING MEN BY

PROFESSOR JOHN PERRY, D.Sc., F.R.S.,

AT

THE MUSEUM OF PRACTICAL GEOLOGY,

JERMYN STREET,

FEBRUARY AND MARCH, 1899,

WITH CERTAIN EXERCISES SUPPOSED TO BE
WORKED AFTER EVERY LECTURE.

LONDON:
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1899.

Price Sixpence.

Nevertheless, there may be some mathematical and didactical points in favour of the rule of false.

Sanford** brings an example of grade 8 children, who are *at the beginning of algebra, who implicitly used the* simple substitution method (without having been taught it), instead of an algebraic solution.

She also brings an interesting historical point. The first use of the symbols + and - were, apparently, in connection with the *double rule of false,* to denote the direction of the error.

Finally, try the following problem, by algebraic methods and by the double rule of false - and compare.

A mother shares out <u>all</u> the sweets she has, between her three sons, as follows: the first gets half of what she has plus 2 sweets, the second gets half of the remainder plus 2 sweets, and the third gets half of what is now left plus 2 sweets. How many sweets did she have altogether?

Maybe there is an even better method than both of the above!

** Sanford, V., The Rule of False Position, *The Mathematics Teacher,* 1951, 44, 307-310.

18. The mathematical operations required in each method are:

Rule of False	*Modern Algebraic Method*
i) Guessing (At this stage it is possible to direct the guessing by estimating the solution, but the method does not require it). ii) Using an algorithm - calculation of the result obtained from the conditions of the problem when the chosen number is substituted. - Rule of three.	i) "Translation" Rewriting of the problem in algebraic language. ii) Using algebraic technique Simplifying the algebraic expression obtained from the "translation" to the form $ax + b = 0$ iii) Using algorithm Solution of $ax + b = 0$

Using the conditions of the problems and the solution in order to check the result.

As can be seen, the operations on the left are arithmetic and on the right algebraic. Students can solve problems (whose algebraic expression is a linear equation in one unknown) before they have learned algebra (as occurred in history).

There is no doubt that the use of an arithmetical algorithm is simpler than translation + algebraic technique. But, on the other hand, the *rule of false position* is likely to be a technique executed without understanding. Such an approach is hardly likely to find much justification today. And if we want to justify the rule, then it is not easy without algebra.

A. Arcavi
M. Bruckheimer

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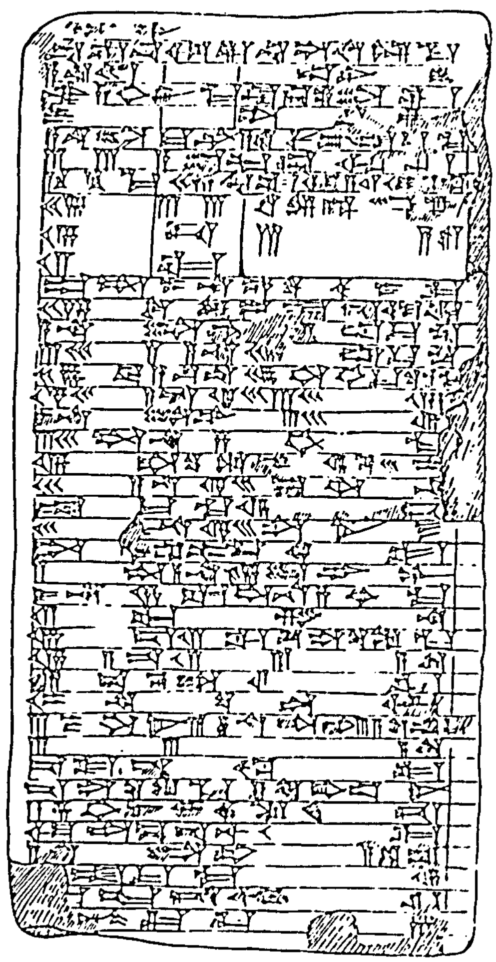
Equations

Worksheet: Babylonian Mathematics



The above illustration is a clay tablet, as it appears in Neugebauer, O. *Mathematische Keilschrifttexte.* Springer, 1935-37.

It shows a problem which is the subject of this worksheet.

Histories of mathematics denote by "Babylonian mathematics", the mathematics of Mesopotamia between 2000 BCE and 300 CE, approximately.

Research into Babylonian mathematics is relatively recent, and most of it belongs to the present century. Today, a considerable amount of information is available as a consequence of the deciphering of Babylonian mathematical documents (clay tablets).

In this worksheet, we shall first look at how the Babylonians "wrote" numbers, and then we shall investigate how they solved certain "quadratic equations".

<u>Cuneiform writing and the sexagesimal system</u>*

The Babylonians wrote numbers in base 60, but they did not have sixty different symbols. They only used two symbols: a wedge 𒁹 , representing units, and a corner 𒌋 , representing tens.

There was, in some sense, a positional system, thus for example

represents $11 \cdot 60^2 + 23 \cdot 60 + 45 = 41025.$

But, since, there was no symbol for zero (at least in the earlier Babylonian period), a given combination of symbols did not determine a number uniquely.

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* This review is based on:
Bunt L.N.H., Jones P.S. & Bedient J.D., *Historical Roots of Elementary Mathematics,* 1976, Prentice Hall.

Thus, for example, 

can represent 12

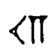
or $10 \cdot 60 + 2 = 602$

or $11 \cdot 60 + 1 = 661$

or $12 \cdot 60 = 720$ etc.

In many cases, the number intended has to be understood from the context. This complicates the deciphering process.

A further difficulty is that there is no symbol to separate the whole from the fractional part of a number.

Thus  , can also represent

$$\frac{12}{60} = \frac{1}{5}$$

or $10 + \frac{2}{60} = 10\frac{1}{30}$ etc.

We shall adopt the accepted form of writing numbers in base 60 using the ten digits 0-9. Thus

a) a comma indicates the different sexagesimal positions.

E.g. 2,30 is $2 \cdot 60 + 30 = 150$.

b) A semicolon separates the whole from the fractional part.

E.g. 0; 1, 2 is $\frac{1}{60} + \frac{2}{60^2} = \frac{31}{1800}$

Questions

1. Fill in the blanks in the following table.

Decimal	Sexagesimal	Babylonian
62	1, 2	-----
----	1, 10	-----
60	-----	-----
----	2, 0, 32	-----
152	-----	-----
$2\frac{1}{2}$	-----	-----
$\frac{5}{6}$	-----	-----

2. Give six different numbers that could be represented by .

3. In later times, the symbol  was introduced, to indicate a zero between two other digits of a number in base 60 (but <u>not</u> at the end of a number). Which ambiguities were removed by the introduction of this symbol and which not?

4. Compare the properties of the Babylonian cuneiform writing with those of Egyptian hieroglyphics.

Solving equations

The following extract* summarizes, in modern notation, the kind of problems in which the Babylonians used "quadratic equations".

The Babylonian Types of the Quadratic Equations

Historically, it would perhaps be more proper to speak of rectangular instead of quadratic equations, because it was the problems of the rectangle that gave rise to these equations. In the square, there is only one unknown quantity, x. If one knows the side, x, one may find the area, x^2, and if one knows the area, he may find the side. In the rectangle, there are two quantities that must be ascertained, the length and the breadth, or the flank and the front, as the Babylonians call them (3), x and y in our designation. If one knows both of them, he may find the area, and if one knows the area and one of the sides, he may find the other side. These are the elementary problems leading to linear equations. But then come the higher, more complicated problems, among which we may distinguish ...

(1) In addition to *the area*, there are given the sum or the difference of the sides ... (2) In addition to *the diagonal*, there are given the sum or the difference of the sides ... (3) In addition to *one side* there are given the sum or the difference of *the diagonal* and the other side ...

Gandz, S. The origin and development of the quadratic equations in Babylonian, Greek and Early Arabic Algebra. *OSIRIS,* 1938, Vol. 3, p. 405-557.

Questions

5. The following table summarizes the kind of problems that according to the extract, the Babylonians solved, and their translation into algebra. Fill in the blanks using x and y for the sides of the rectangle.

Type of problem	In algebraic notation
Given	
I) the area and the sum of the sides	----- -----
II) the area and the difference between the sides	----- -----
III) the diagonal and the sum of the sides	----- -----
IV) the diagonal and the difference between the sides	----- -----
V) one side and the sum of the diagonal with the other side	----- -----
VI) one side and the difference between the diagonal and the other side	----- -----

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Neugebauer*, one of the researchers of Babylonian mathematics, notes that types I and II are the most common.

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* Neugebauer O., *The Exact Sciences in Antiquity* Dover, 1969, p. 41.

He writes:

> ... **the main type of quadratic problems of which we have hundreds of examples preserved, a type which I call "normal form": two numbers should be found if (a) their product and (b) their sum or difference is given. It is obviously the purpose of countless examples to teach the transformation of more complicated quadratic problems to this "normal form" ...**

The following extract deals with the solution of a problem of the "normal form". The extract* is a part of a larger problem (to which we shall refer later) from the earlier period (1700 BCE approximately).

The following problem is being solved: *the sum of the sides of a rectangle is 29, their product 210, find the side lengths.*

Take one half of 29 (this gives 14;30).

14;30 × 14;30 = 3,30;15

3,30;15 – 3,30 = 0;15.

The square root of 0;15 is 0;30.

14;30 + 0;30 = 15 length

14;30 – 0;30 = 14 width.

Questions

6. Rewrite the solution in base 10.

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* In the English version, as it appears in Van der Waerden, B.L. *Science Awakening* Noordhoff Ltd., 1954, p. 63-65.

7. Apparently the Babylonians assumed the length to be $\frac{29}{2} + c$ (i.e., half the sum plus an unknown number) and the width as $\frac{29}{2} - c$.

a) Why is the assumption legitimate?

b) Substitute $\frac{29}{2} + c$ and $\frac{29}{2} - c$ in the expression for the product of the sides, and calculate c.

c) Using c, calculate the length and the width.

8. Generalise algebraically, the way the Babylonians solved a problem of the form

$$xy = a$$
$$x + y = b \ .$$

9. How would you solve this problem today?

The following extract brings the full problem from which the above was part.

Length, width. I have multiplied length and width, thus obtaining the area. Then I added to the area, the excess of the length over the width: 3,3 (i.e. 183 was the result). Moreover, I have added length and width: 27. Required length, width and area.

(given:) 27 and 3,3, the sums
(result:) 15 length 3,0 area.
12 width

One follows this method:

$$27 + 3,3 = 3,30$$
$$2 + 27 = 29.$$

Take one half of 29 (this gives 14;30).

$$14;30 \times 14;30 = 3,30;15$$
$$3,30;15 - 3,30 = 0;15.$$

The square root of 0;15 is 0;30.

$$14;30 + 0;30 = 15 \text{ length}$$
$$14;30 - 0;30 = 14 \text{ width.}$$

Subtract 2, which has been added to 27, from 14, the width. 12 is the actual width. I have multiplied 15 length by 12 width.

$$15 \times 12 = 3,0 \text{ area.}$$
$$15 - 12 = 3$$
$$3,0 + 3 = 3,3.$$

<u>Questions</u>

10. a) Rewrite the problem in modern terms.
    b) Rewrite the solution in base 10.

11. Divide the solution into stages, translate each stage into algebraic expressions, and explain what is done in each of them.

12. Solve the problem as it is usually solved today, and compare it with the Babylonian solution.

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Equations

<u>Answer sheet</u>: Babylonian Mathematics

1.

Decimal	Sexagesimal	Babylonian
62	1,2	𒁹 𒈫
$2\cdot 60 + 10 = 130$	2,10	𒈫 𒌋
60	1,0	𒁹
$2\cdot 60^2 + 32 = 3632$	2,0,32	𒈫 𒌍𒈫
152	2,32	𒈫 𒌍𒈫
$2\frac{1}{2}$	2;30	𒈫 𒌍
$\frac{5}{6}$	0;50	𒐐

2. 𒌋𒁹 can represent

11, 660, 601, 3601, $\frac{11}{60}$, $1\frac{1}{10}$ etc.

3. For instance, the two numbers indicated in the table above could be distinguished as follows.

𒈫 𒑱 𒌍𒈫 (3632) 𒈫𒌍𒈫 (152)

But, for instance 𒌋𒁹 still represents both 11 and 660.

4.

Hieroglyphics	Cuneiform
- base 10	- base 60
- 7 different symbols for 1, 10, ..., 1,000,000	- 2 different symbols for units and tens
- there is no place value	- place value (only partially)

5.

Type of problem	In algebraic notation
Given	
I) the area and	$x \cdot y = a$
the sum of the sides	$x + y = b$
II) the area and	$x \cdot y = a$
the difference of the sides	$x - y = b$
III) the diagonal and	$x^2 + y^2 = a$
the sum of the sides	$x + y = b$
IV) the diagonal and	$x^2 + y^2 = a$
the difference of the sides	$x - y = b$
V) One side and	$x^2 - y^2 = a$ *
the sum of the diagonal with the other side	$x + y = b$
VI) One side and	$x^2 - y^2 = a$
the difference between the diagonal and the other side	$x - y = b$

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* In order to have a uniform representation, in this case x represents the diagonal and not the side.

6. Take one half of 29 (this gives 14.5).

$$14.5 \times 14.5 = 210.25$$

$$210.25 - 210 = 0.25$$

The square root of 0.25 is 0.5

$$14.5 + 0.5 = 15 \quad \text{length}$$

$$14.5 - 0.5 = 14 \quad \text{width}$$

7. a) It is legitimate (though unusual) to think of $\frac{29}{2} + c$ as the length and then the width is $29 - (\frac{29}{2} + c) = \frac{29}{2} - c$.

b)
$$(\tfrac{29}{2} + c)\cdot(\tfrac{29}{2} - c) = 210$$
$$210.25 - c^2 = 210$$
$$c^2 = 0.25$$
$$c = 0.5$$

c) The length $\quad 14.5 + 0.5 = 15$
The width $\quad 14.5 - 0.5 = 14$

If we were to take into account the negative root, i.e. $c = -0.5$, we would obtain the same mathematical answer for the sides of the rectangle, but 14 for the "length" and 15 for the "width".

The following extract* characterizes the way in which the Babylonians solved problems of this type.

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*Taken from,
Bunt L.N.H., Jones P.S. & Bedient J.D.
Historical Roots of Elementary Mathematics, 1976, Prentice Hall.

The Babylonians followed the same line of thought as we do, but did not have anything approximating modern notation. They could therefore only express their procedures by means of numerical examples. That is why we find in their texts a great many problems that are all solved in the same way. The solutions nearly always terminate with the statement: "such is the procedure." Thus, it appears that these problems were intended to demonstrate a general method of solution.

8. $$x \cdot y = a \qquad (1)$$
$$x + y = b \qquad (2)$$

so if in (2) we write

$$x = \frac{b}{2} + c$$

then $$y = \frac{b}{2} - c\ .$$

Hence substituting in (1), we obtain

$$\left(\frac{b}{2} + c\right)\cdot\left(\frac{b}{2} - c\right) = a$$

$$\frac{b^2}{4} - c^2 = a$$

$$c^2 = \frac{b^2}{4} - a$$

$$c = \sqrt{\frac{b^2}{4} - a}$$

$$x = \frac{b}{2} + \sqrt{\frac{b^2}{4} - a}$$

$$y = \frac{b}{2} - \sqrt{\frac{b^2}{4} - a}$$

Historians seem to agree with the assumption that this is algebraic expression for the way the Babylonians solved problems of this type.

But, if we look at the operations performed it is possible to suggest another algebraic procedure.

Babylonians	Usual interpretation	Other interpretation
Take one half of 29 (this gives 14.5)	$x + y = 29$ length $x = \frac{29}{2} + c$ width $y = \frac{29}{2} - c$	$x + y = 29$ $\frac{x+y}{2} = \frac{29}{2}$
14.5 x 14.5 = 210.25	$(14.5+c)\cdot(14.5-c) = 210.25$	$(\frac{x+y}{2})^2 = 210.25$
210.25 - 210 = 0.25	since $x\cdot y = 210$ then $210.25 - c^2 = 210$ thus $c^2 = 0.25$	$(\frac{x+y}{2})^2 - xy = 210.25-210$ $(\frac{x-y}{2})^2 = 0.25$
The square root of 0.25 is 0.5	$c = 0.5$	$\frac{x-y}{2} = 0.5$
length 14.5 + 0.5 = 15	length $14.5 + c = 14.5 + 0.5 = 15$	$\frac{x+y}{2} + \frac{x-y}{2} = 14.5 + 0.5$ $x = 15$
width 14.5 - 0.5 = 15	width $14.5 - c = 14.5 - 0.5 = 14$	$\frac{x+y}{2} - \frac{x-y}{2} = 14.5 - 0.5$ $y = 14$

9. For example, we can solve this as follows.

$$x \cdot y = a$$
$$x + y = b$$
$$y = b - x$$
$$x \cdot (b - x) = a$$
$$xb - x^2 = a$$
$$x^2 - xb + a = 0$$

and the solution of this equation is

$$\frac{b \pm \sqrt{b^2 - 4a}}{2}$$

10. a) If we add the area of the rectangle to the difference between the sides, we obtain 183. The sum of the sides is 27. Calculate the length, the width and the area.

b) (Given) 27 and 183 the sums

(result) 15 length, 180 area

12 width

One follows this method:

27 + 183 = 210

2 + 27 = 29

Take one half of 29 (this gives 14.5)

14.5 x 14.5 = 210.25

210.25 - 210 = 0.25

The square root of 0.25 is 0.5

$$14.5 + 0.5 = 15 \text{ length}$$

$$14.5 - 0.5 = 14 \text{ width}$$

Subtract 2, which has been added to 27, from 14 the width. 12 is the actual width. I have multiplied 15 length by 12 width.

$$15 \times 12 = 180 \text{ area}$$
$$15 - 12 = 3$$
$$180 + 3 = 183$$

11. The problem is :

$$x \cdot y + x - y = 183$$
$$x + y = 27$$

Calculation with numbers	Corresponding algebra
$27 + 183 = 210$	$xy + x - y + x + y = 183 + 27$
	$xy + 2x = 210$
	$x(y + 2) = 210$
$2 + 27 = 29$	$x + y + 2 = 27 + 2$
	$x + (y + 2) = 29$
	let's take $z = y + 2$

Apparently, up to this point, the problem was brought to its "normal form", by means of a change of variables.

Now the problem is

$$x \cdot z = 210$$

$$x + z = 29$$

And this is the system to be solved from here onwards.

In a further stage, the original variable y is sought,

$$\text{since} \quad z = 14$$

$$y = 12$$

The last stage consists of checking the solutions.

12.

$$xy + x - y = 183$$

$$x + y = 27$$

$$y = 27 - x$$

We substitute $x(27 - x) + x - 27 + x = 183$

and obtain $x^2 - 29x + 210 = 0$

Whence $x_1 = 15 \qquad x_2 = 14$

for which $y_1 = 12 \qquad y_2 = 13$ respectively.

As can be seen, the Babylonians gave only one solution.

The following extract* defines the special character of the problems from Babylonian mathematics that we have met in this worksheet.

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* Taken from Bunt, Jones & Bedient, ibid.

The Babylonians were not used to working with abstract numbers as we are. They therefore talked of quantities, of what we sometimes term *denominate numbers*, numbers associated with units of measure (in the example: length, width, and area). Yet it does not appear that they had formulated a real geometrical problem, for in that case they probably would not have added length to area. Terms such as "length" only served to give a name to the unknown numbers, a process similar to what we often do in our arithmetic problems.

The preceding examples illustrate the fact that the Babylonians were aware of connections between geometry and algebra. Geometric terminology was added to give concreteness to the solution of an algebraic problem. However, their procedure of subtracting a length from an area shows that they had no objections to mixing dimensions.

In later centuries there was reluctance to mix "dimensions," that is, to combine numbers representing areas with those representing lengths or volumes. This reluctance is found from the Greek period through the work of the Italian algebraists of the sixteenth century and even up to the time of *Descartes* (1596-1650). Descartes, in his development of analytic geometry and the related theory of equations, did write and deal with expressions such as $3x^4 - 4x^3 + 5x^2 - 6x = 7$, where terms were combined even though their geometric counterparts would have been "squares" and "cubes," which are measured in different units. Although the development of the theory of equations was often helped by the use of geometric diagrams and concepts, it also was hindered by too-close ties to geometry.

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Equations

Worksheet: Euclid and *The Elements*

Very little is known about Euclid. He lived about the year 300 B.C.E. and apparently taught in Alexandria. He is famous as the author of the *Elements,* arguably the best known book in the history of mathematics. Since the invention of printing there have been numerous editions in many languages - and many adaptations. A standard English edition is that of Heath*, who added a scholarly commentary. The following extracts are to set the scene. The first extract is taken from Heath's introduction.

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* Heath T.L., *The Thirteen Books of Euclid's Elements Translated with Introduction and Commentary.* Dover Pub., N.Y., 1956.

As in the case of the other great mathematicians of Greece, so in Euclid's case, we have only the most meagre particulars of the life and personality of the man.

Most of what we have is contained in the passage of Proclus' summary relating to him, which is as follows : *

. . . Euclid, who put together the Elements, collecting many of Eudoxus' theorems, perfecting many of Theaetetus', and also bringing to irrefragable demonstration the things which were only somewhat loosely proved by his predecessors. . . . **

. . . It is most probable that Euclid received his mathematical training in Athens from the pupils of Plato; for most of the geometers who could have taught him were of that school, and it was in Athens that the older writers of elements, and the other mathematicians on whose works Euclid's *Elements* depend, had lived and taught.

From Heath***

This wonderful book, with all its imperfections, which indeed are slight enough when account is taken of the date at which it appeared, is and will doubtless remain the greatest mathematical text-book of all time. . . .

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* Proclus (410-485 C.E.) is one of the commentators through whom many details of Greek mathematics have been preserved.

** To Eudoxus and Theaetetus are attributed many ideas which are brought in the *Elements*, in particular those in Book X, from which we quote later.

*** Heath T.L., *Greek Mathematics*, Vol. I, Oxford Univ. Press, 1921, p. 357-358.

From Boyer*

The *Elements* is divided into thirteen books or chapters, of which the first half dozen are on elementary plane geometry, the next three on the theory of numbers, Book X on incommensurables, and the last three chiefly on solid geometry. There is no introduction or preamble to the work, and the first book opens abruptly with a list of twenty-three definitions. The weakness here is that some of the definitions do not define, inasmuch as there is no prior set of undefined elements in terms of which to define the others. Thus to say, as does Euclid, that "a point is that which has no part," or that "a line is breadthless length," or that "a surface is that which has length and breadth only," is scarcely to define these entities, for a definition must be expressed in terms of things that precede, and are better known than the things defined. ...

Following the definitions, Euclid lists five postulates and five common notions. Aristotle had made a sharp distinction between axioms (or common notions) and postulates; the former, he said, must be convincing in themselves -- truths common to all studies - but the latter are less obvious and do not presuppose the assent of the learner, for they pertain only to the subject at hand ...

... In its time the *Elements* evidently was the most tightly reasoned logical development of elementary mathematics that had ever been put together, and two thousand years were to pass before a more careful presentation occurred. During this long interval most mathematicians regarded the treatment as logically satisfying and pedagogically sound.

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* Boyer C.B., *A History of Mathematics*, J. Wiley & S., 1968, p. 115-118.

Question

1. The extract from Boyer distinguishes between definitions, postulates and common notions (axioms). Decide which of these three is the appropriate description for each of the following statements*.

   a) There exists only one straight line containing two given points.

   b) The whole is greater than the part.

   c) Things which are equal to the same thing are also equal to one another.

   d) All right angles are equal to one another.

   e) An obtuse is an angle greater than a right angle.

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* The sentences are taken from Heath's English version of the *Elements*.

Book II of Euclid's *Elements* is often regarded as dealing with geometrical algebra, although the algebra and its notation as we know it, appeared some 1800 years after Euclid. In the following extract Boyer (ibid) discusses this point. (The omission is for the purpose of Qu. 3b.)

> It is sometimes asserted that the Greeks had no algebra, but this is patently false. They had Book II of the *Elements*, which is a geometrical algebra that served much the same purpose as does our symbolic algebra. There can be little doubt that modern algebra greatly facilitates the manipulation of relationships among magnitudes. But it is undoubtedly also true that a Greek geometer versed in the fourteen theorems of Euclid's "algebra" was far more adept in applying these theorems to practical mensuration than is an experienced geometer of today. Ancient geometrical algebra was not an ideal tool, but it was far from ineffective. Euclid's statement (Proposition 4), "If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments," is a verbose way of saying that _ _ _ _ _ _ _ _ _ _ _ , but its visual appeal to an Alexandrian schoolboy must have been far more vivid than its modern algebraic counterpart can ever be. True, the proof in the *Elements* occupies about a page and a half; but how many high school students of today could give a careful proof of the algebraic rule they apply so unhesitatingly?

Questions

2. What does Euclid mean by
   "the square on the whole"
   "the squares on the segments"
   "the rectangle contained by the segments"

3. a) Draw an appropriate picture for Proposition 4.

   b) Write down the algebraic formula (missing in the text) that corresponds to the "verbose way of saying" Proposition 4.

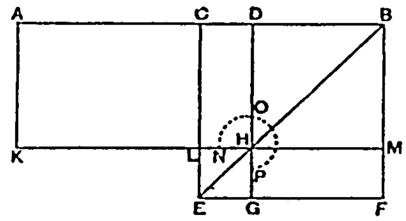
The following is Proposition 5 from Book II*.

PROPOSITION 5.

If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half.

For let a straight line *AB* be cut into equal segments at *C* and into unequal segments at *D*;

I say that the rectangle contained by *AD*, *DB* together with the square on *CD* is equal to the square on *CB*.



Questions

4. Identify in the drawing the following figures
   "the rectangle contained by AD, DB"
   "the square on CD"
   "the square on CB"

5. a) Rewrite Proposition 5 in terms of areas and check visually if it looks right.
   b) Rewrite and prove Proposition 5 algebraically, taking AB = a and AD = x.

In the following extract, Heath (ibid) make some comments on another possible algebraic interpretation of Proposition 5.

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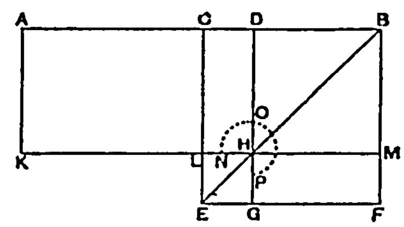
* At it appears in :

Heath T.L. *The thirteen books of Euclid's Elements.* Translated with introduction and commentary. Dover Pub., 1956, p. 383-4.

The ommissions are for the purpose of the following exercises which will consist in deciphering Heath's algebraic interpretation (Qu. 6 and 7).



Suppose, in the figure of II. 5, that $AB = a$, $DB = x$;

then _ _ _ _ _ = the rectangle AH

= the gnomon NOP.

Thus, if the area of the gnomon is given ($= b^2$, say), and if a is given ($= AB$), the problem of solving the equation _ _ _ _ _ _ _

is, in the language of geometry, *To a given straight line* (a) *to apply a rectangle which shall be equal to a given square* (b^2) *and shall fall short by a square figure,* i.e. to construct the rectangle AH or the gnomon NOP.

Questions

6. What is the "gnomon NOP"?

7. Write down the two missing formulae according to Proposition 5 and the initial suppositions.

8. Another sort of algebraic expression can be obtained from Proposition 5, if one takes AC = a and CD = b. Find it.

9. What are the corresponding algebraic expressions if AD = x, DB = y and if the segment AB and the area of the rectangle "contained by AD, DB" are given?

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Equations

Answer sheet: Euclid and the *Elements*

1. a) and d) are postulates, since they are related to the subject matter.

   b) and c) are axioms - "truths common to all studies". (Note, however, that whereas c) might be described to be universal, b) is not. Thus, in modern times, an infinite set was defined as a set which can be put into one-one correspondence with a subset of itself. For example,

$$\begin{array}{ccccccccc} 1, & 2, & 3, & 4, & 5, & \ldots\ldots, & n, & \ldots\ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \\ 2, & 4, & 6, & 8, & 10, & \ldots\ldots, & 2n, & \ldots\ldots \end{array}$$

   And, in this sense, the "whole is not greater than its part".) Finally e) is a definition of a new term. (Notice also that d) is a postulate logically linked to the axiom in c).)

   In the first book of the *Elements,* Euclid brings 23 definitions (point, line, circle, etc.), 5 postulates and 5 axioms.

   In the following we bring the 5 postulates as they appear in the Boyer* version.

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*See Boyer C.B., *A History of Mathematics,* Wiley, 1968, p. 116-7.

Postulates.

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any center and radius.
4. That all right angles are equal.
5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than the two right angles.

The fifth postulate is distinguished from the rest by its complicated enunciation, and it was the most problematic of the five. For some 2000 years, mathematicians tried to prove it and/or to propose alternatives (Heath lists nine), such as

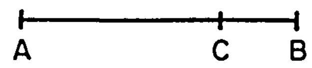
- through a given point, there is just one straight line parallel to a given straight line.
- if a straight line cuts one of two parallel straight lines, then it also cuts the second.

In the end, in the nineteenth century there was an interesting and surprising development. The mathematicians Gauss (1777-1855), Bolyai (1802-1860) and Lobachevskii (1792-1856) showed that it was possible to construct a consistent geometry, and not only one, in which the fifth postulate was negated. Such geometries are called *non-Euclidean*. The existence of such geometries shows that postulate 5 cannot be proved.

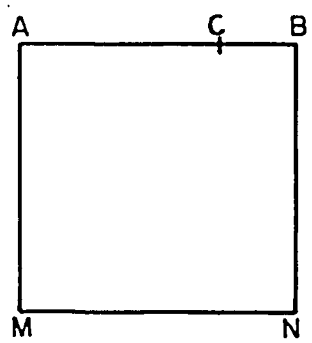
2. "The square on the whole" is the square that has the given "straight line" as its side.
   "The squares on the segments" are the squares that have each of the respective segments as sides.
   "The rectangle contained by the segments" is the rectangle whose length is the larger segment of the given "straight line" and whose width is the remaining segment.

3. a)

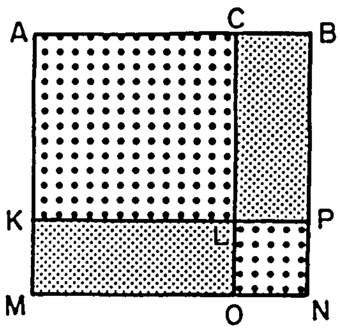
"If a straight line (AB) be cut at random (at C say),



the square on the whole (ABMN, say)



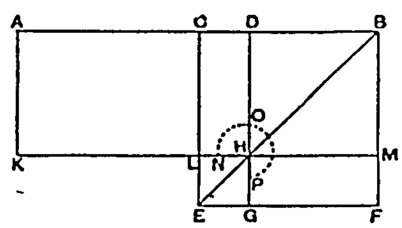
is equal to the squares on the segments (ACKL and LPON, say) and twice the rectangle contained by the segments (KLMO)."



b) As can be seen from the drawing, if AC=a and CB=b, then the corresponding algebraic formula is

$$(a + b)^2 = a^2 + b^2 + 2ab$$

4.



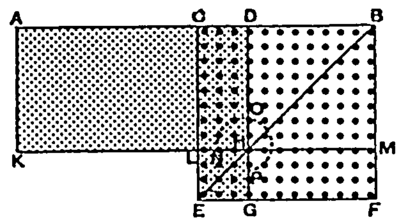
"The rectangle contained by AD, DB" is the rectangle ADHK, since DH = DB.

"The square on CD" is the square LHGE.

"The square on CB" is the square CBFE.

5. a) In terms of areas, the proposition says that the area of the rectangle ADHK, is less than the area of the square CBFE by the square LHGE.

Visually,



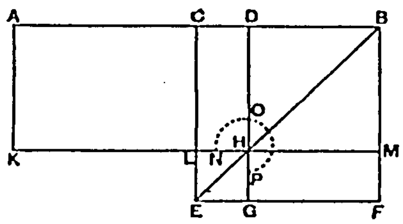
i.e., if we remove the area CDHL from both the rectangle ADHK and the square CBFE and, in addition, if we remove also the little square LHGE from the latter, the remainders are equal areas (ACLK and DBFG).

b) If AB = a
and AD = x

$$x \cdot (a - x) + (x - \frac{1}{2} a)^2 = (\frac{1}{2} a)^2 .$$

Elementary algebraic simplification of the left-hand side, proves the identity.

Compare it with the geometrical proof in Euclid.



For let the square *CEFB* be described on *CB*, [I. 46]
and let *BE* be joined;
through *D* let *DG* be drawn parallel to either *CE* or *BF*,
through *H* again let *KM* be drawn parallel to either *AB* or *EF*,
and again through *A* let *AK* be drawn parallel to either *CL* or *BM*. [I. 31]

Then, since the complement *CH* is equal to the complement *HF*, [I. 43]
let *DM* be added to each;
therefore the whole *CM* is equal to the whole *DF*.
But *CM* is equal to *AL*,
since *AC* is also equal to *CB*; [I. 36]
therefore *AL* is also equal to *DF*.
Let *CH* be added to each;
therefore the whole *AH* is equal to the gnomon *NOP*.

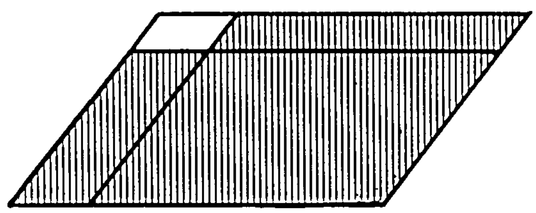
But *AH* is the rectangle *AD*, *DB*, for *DH* is equal to *DB*,
therefore the gnomon *NOP* is also equal to the rectangle *AD*, *DB*.

Let *LG*, which is equal to the square on *CD*, be added to each;
therefore the gnomon *NOP* and *LG* are equal to the rectangle contained by *AD*, *DB* and the square on *CD*.

But the gnomon *NOP* and *LG* are the whole square *CEFB*, which is described on *CB*;
therefore the rectangle contained by *AD*, *DB* together with the square on *CD* is equal to the square on *CB*.

The square brackets indicate propositions previously demonstrated on which the arguments are based.

6. From the context, we see that the gnomon NOP is the square CBFE from which the little square LHGE has been removed. In general, gnomon is the remainder of a parallelogram after the removal of a similar parallelogram as illustrated.



The shaded area is gnomon.

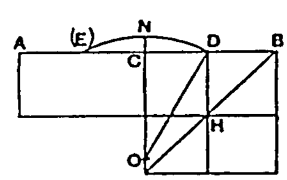
7. $x \cdot (a - x)$ = the rectangle ADHK

Thus if a (the length of the segment) and b^2 are given, then the problem of finding the point D is equivalent to solving the quadratic equation

$$ax - x^2 = b^2.$$

In the following extract, Heath (ibid) describes the geometrical way of finding this point.

Draw *CO* perpendicular to *AB* and equal to *b*; produce *OC* to *N* so that $ON = CB$ (or $\frac{1}{2}a$); and with *O* as centre and radius *ON* describe a circle cutting *CB* in *D*.



Then *DB* (or *x*) is found, and therefore the required rectangle *AH*.

For the rectangle *AD*, *DB* together with the square on *CD* is equal to the square on *CB*, [II. 5]

i.e. to the square on *OD*,

i.e. to the squares on *OC*, *CD*; [I. 47]

whence the rectangle *AD*, *DB* is equal to the square on *OC*,

or $$ax - x^2 = b^2.$$

It is of course a necessary condition of the possibility of a real solution that b^2 must not be greater that $(\frac{1}{2}a)^2$. This condition itself can easily be obtained from Euclid's proposition; for, since the sum of the rectangle *AD*, *DB* and the square on *CD* is equal to the square on *CB*, which is constant, it follows that, as *CD* diminishes, i.e. as *D* moves nearer to *C*, the rectangle *AD*, *DB* increases and, when *D* actually coincides with *C*, so that *CD* vanishes, the rectangle *AD*, *DB* becomes the rectangle *AC*, *CB*, i.e. the square on *CB*, and is a maximum.

8. $(a + b)\cdot(a - b) = a^2 - b^2$

9. If we denote $AB = a$
and the area $ADHK = b^2$
then

$$x + y = a$$
$$x \cdot y = b^2$$

which is, as we saw in the previous worksheet, the "the normal form" of the Babylonian type of equations.

Here also we can visualize their change of variable

$$x = \frac{a}{2} + z$$

$$y = \frac{a}{2} - z$$

where z is CD.

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Equations

Worksheet: Al-Khowarizmi

The second half of the 8th century and the 9th century C.E. scientific and cultural activity flourished in the Baqdad Caliphate. The Arabs translated many of the Greek Mathematical works. This proved to be very important, since much of what is known today about Greek mathematics, reached us via these Arabic sources.

But the Arabic contribution is not confined to the preservation and transfer of other cultures. They made important contributions of their own. Mohammed-ibn Musa al-Khowarizmi was one of the famous mathematicians of this period.

Little is known about his life, besides the fact that he lived in the first half of the 9th century and wrote a number of books on mathematics. One of the most important is *Al-jabr wa'l muqabalah*. From the title of this book we have the word *algebra*, and from the author's name the word *algorithm*. In the following we quote some extracts from this work, in the English version, which is a translation and adaptation from the Latin version*.

---

* Taken from,

Karpinski, L.C. Robert of Chester's Latin Translation of the Algebra of Al-Khowarizmi, 1915.

CHAPTER I

Concerning squares equal to roots

The following is an example of squares equal to roots: a square is equal to 5 roots. The root of the square then is 5, and 25 forms its square which, of course, equals five of its roots.

Another example: the third part of a square equals four roots. Then the root of the square is 12 and 144 designates its square.

<u>Questions</u>

1. Complete the following table.

According to the text	Modern
------------	x^2
root	------
a square is equal to five roots	------
-----------------------	$\frac{x^3}{3} = 4x$

2. What solutions does Al-Khowarizmi gives for "the third part of a square equals four roots"?

   What are the solutions that you would give?

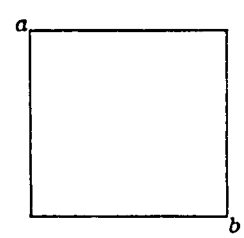
CHAPTER IV

Concerning squares and roots equal to numbers

The following is an example of squares and roots equal to numbers: a square and 10 roots are equal to 39 units. The question therefore in this type of equation is about as follows: what is the square which combined with ten of its roots will give a sum total of 39? The manner of solving this type of equation is to take one-half of the roots just mentioned. Now the roots in the problem before us are 10. Therefore take 5, which multiplied by itself gives 25, an amount which you add to 39, giving 64. Having taken then the square root of this which is 8, subtract from it the half of the roots, 5, leaving 3. The number three therefore represents one root of this square, which itself, of course, is 9. Nine therefore gives that square.

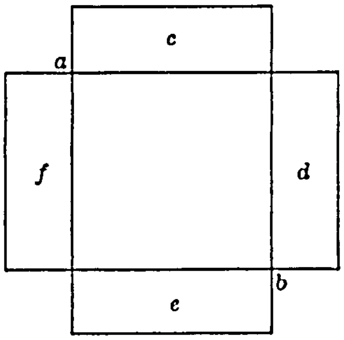
GEOMETRICAL DEMONSTRATIONS

...it is necessary



that we should demonstrate geometrically the truth of the same problems which we have explained in numbers. Therefore our first proposition is this, that a square and 10 roots equal 39 units.

The proof is that we construct a square of unknown sides, and let this square figure represent the square (second power of the unknown) which together with its root you wish to find. ...



Since then ten roots were proposed with the square, we take a fourth part of the number ten and apply to each side of the square an area ... of which the length should be the same as the length of the square first described and the breadth $2\frac{1}{2}$, which is a fourth part of 10. ...

The size of the areas in each of the four corners, which is found by multiplying $2\frac{1}{2}$ by $2\frac{1}{2}$, completes that which is lacking ...

Questions

3. Write "a square and 10 roots are equal to 39 units" in algebraic form and solve.

4. a) Rewrite the solution of the equation, following the text step by step.

   b) Rewrite, in general, (using parameters) the solution of the equation, following the text step by step.

5. Indicate in the drawings what is represented by the square ab, and the rectangles c, d, e, f.

6. Complete the geometrical solution following the numerical solution.

7. Can you think of a different geometrical solution?

8. The following is another problem from Al-Khowarizmi's book: "a square and ten roots equal to 56 units".

   a) Solve numerically by Al-Khowarizmi's method.

   b) Solve geometrically.

   c) Solve the equation as you would solve it today.

CHAPTER V

Concerning squares and numbers equal to roots

The following is an illustration of this type: a square and 21 units equal 10 roots. The rule for the investigation of this type of equation is as follows: what is the square which is such that when you add 21 units the sum total equals 10 roots of that square? The solution of this type of problem is obtained in the following manner. You take first one-half of the roots, giving in this instance 5, which multiplied by itself gives 25. From 25 subtract the 21 units...

This gives 4, of which you extract the square root, which is 2. From the half of the roots, or 5, you take 2 away, and 3 remains, constituting one root of this square which itself is, of course, 9.

If you wish you may add to the half of the roots, namely 5, the same 2 which you have just subtracted from the half of the roots. This give 7, which stands for one root of the square, and 49 completes the square. Therefore when any problem of this type is proposed to you, try the solution of it by addition as we have said. If you do not solve it by addition, without doubt you will find it by subtraction. And indeed this type alone requires both addition and subtraction, and this you do not find at all in the preceding types.

You ought to understand also that when you take the half of the roots in this form of equation and then multiply the half by itself, if that which proceeds or results from the multiplication is less than the units abovementioned as accompanying the square, you have no equation. If equal to the units, it follows that a root of the square will be the same as the half of the roots which accompany the square, without either addition or diminution.

Questions

9. Rewrite the equation in the paragraph in algebraic form and solve it.
10. Rewrite in general (using parameters) Al-Khowarizmi's process of solution, step by step.
11. What is the difference between this equation and the equation in the previous paragraph.
12. To which case does Al-Khowarizmi refer when he says: "You have no equation"?
13. In which case do we obtain the root "without either addition or diminution"?

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Equations

Answer sheet: Al-Khowarizmi

1.

According to the text	Modern
a square	x^2
root	x
a square is equal to five roots	$x^2 = 5x$
the third part of a square equals four roots	$\frac{x^2}{3} = 4x$

G.H.F. Nesselmann in his book *Die Algebra der Griechen* (Berlin, 1842) distinguishes between three different periods in the history of algebra: the rhetorical, the syncopated and the symbolic. In rhetorical algebra all was written in words, and this is the case in Al-Khowarizmi's book, as we see in the extract.

2. For the equation "the third part of a square equals four roots" Al-Khowarizmi gives 12 as the root, omitting the second possible root which is zero. In fact, he deals with positive roots of equations only.

3. $x^2 + 10x = 39$

$$x = \frac{-10 \pm \sqrt{100 + 156}}{2}$$

$x_1 = 3$

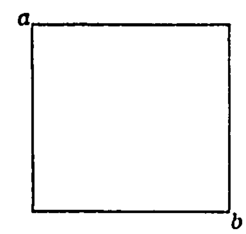
$x_2 = -13$

4.

In words	a) numerically	b) in general
take one half of the roots	5	if the equation is $x^2 + bx = c$ then take $\frac{b}{2}$
multiplied by itself	25	$(\frac{b}{2})^2$
which you add to 39 giving 64	25 + 39 = 64	$(\frac{b}{2})^2 + c$
having taken the square root	8	$\sqrt{(\frac{b}{2})^2 + c}$
subtract from it half of the roots	8 - 5 = 3	$-\frac{b}{2} \div \sqrt{(\frac{b}{2})^2 + c}$

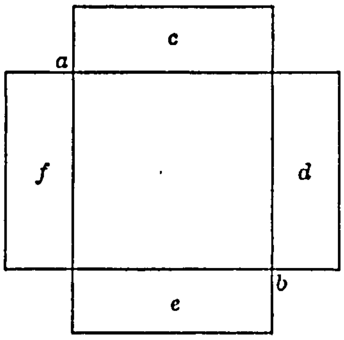
GEOMETRICAL DEMONSTRATIONS

... it is necessary that we should demonstrate geometrically the truth of the same problems which we have explained in numbers. Therefore our first proposition is this, that a square and 10 roots equal 39 units.



The proof is that we construct a square of unknown sides, and let this square figure represent the square (second power of the unknown) which together with its root you wish to find. ...

Since then ten roots were proposed with the square, we take a fourth part of the number ten and apply to each side of the square an area ... of which the length should be the same as the length of the square first described and the breadth $2\frac{1}{2}$, which is a fourth part of 10. ...



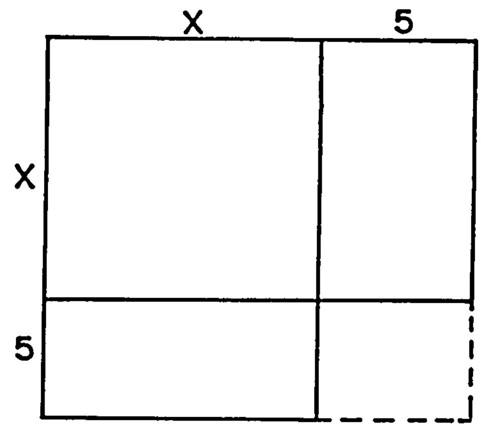
The size of the areas in each of the four corners, which is found by multiplying $2\frac{1}{2}$ by $2\frac{1}{2}$, completes that which is lacking ...

The square ab represents x^2, to which 10x is added in the form of 4 rectangles whose areas are 2.5x each.

If we add 25, in fact we are adding the four missing corners in order to obtain a new square whose area is now 64. Thus its side is 8, from which we have to subtract 5 $(2\frac{1}{2} + 2\frac{1}{2})$ in order to obtain the original side.

Solving the problem geometrically, it is clear that the negative solution will not be found.

7. A different geometrical solution could be, for example,



8. a) The following is the solution as it appears in the original.

Another possible example: half a square and five roots are equal to 28 units.[3] The import of this problem is something like this: what is the square which is such that when to its half you add five of its roots the sum total amounts to 28? Now however it is necessary that the square, which here is given as less than a whole square, should be completed.[4] Therefore the half of this square together with the roots which accompany it must be doubled. We have then, a square and 10 roots equal to 56 units. Therefore take one-half of the roots, giving 5, which multiplied by itself produces 25. Add this to 56, making 81. Extract the square root of this total, which gives 9, and from this subtract half of the roots, 5, leaving 4 as the root of the square.

In this manner you should perform the same operation upon all squares, however many of them there are, and also upon the roots and the units.

b) The geometrical solutions are similar to those described in Qu. 6 and 7.

c) $x^2 + 10x = 56$

$$x = \frac{-10 \pm \sqrt{100 + 224}}{2}$$

$x_1 = 4 \qquad x_2 = -14$

Here again the positive root only corresponds to the solution given by Al-Khowarizmi.

9. $x^2 + 21 = 10x$

$$x = \frac{10 \pm \sqrt{100 - 84}}{2}$$

$x_1 = 7 \qquad x_2 = 3$

Here the two roots are positive, so they are both considered.

10. $x^2 + c = bx$

<u>In the extract</u>	<u>In general, today</u>
take one half of the roots	$\frac{b}{2}$
multiplied by itself	$(\frac{b}{2})^2$
subtract the units	$(\frac{b}{2})^2 - c$
extract the square root	$\sqrt{(\frac{b}{2})^2 - c}$
from the half of the roots you take away	$\frac{b}{2} - \sqrt{(\frac{b}{2})^2 - c}$
if you wish you may add to the half of the roots	$\frac{b}{2} + \sqrt{(\frac{b}{2})^2 - c}$

11. In this paragraph, as we saw, the equation is of the form

$$x^2 + c = bx$$

in contrast to the previous form

$$x^2 + bx = c$$

It can be seen that the former admits two (positive) roots, while the latter admits only one (positive) root. (See the Frend worksheet in the sequence on negative numbers).

12. Al-Khowarizmi says:

> You ought to understand also that when you take the half of the roots in this form of equation and then multiply the half by itself, if that which proceeds or results from the multiplication is less than the units above-mentioned as accompanying the square, you have no equation.'

which in our words means that $(\frac{b}{2})^2 < c$.
This is the case of no real roots.

13. Al-Khowarizmi continues as follows:

> If equal to the units, it follows that a root of the square will be the same as the half of the roots which accompany the square, without either addition or diminution.

which in our words means $(\frac{b}{2})^2 = c$.
This is the case of one root.

Al-Khowarizmi also brings a geometrical solution for the above case ("a square and units equal to roots"), which is a bit more complicated than the previous.

Concerning a Square and Units Equal to Unknown Quantities

A square and 21 units are equal to ten unknowns. This proposition or problem was proposed in the fifth chapter and here a geometrical demonstration is presented.

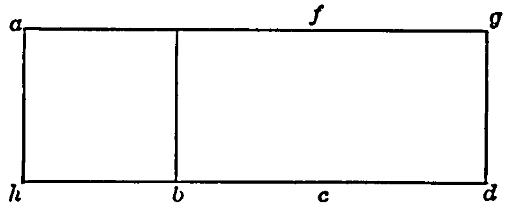


Fig. 7. — Incomplete figure. From the Columbia manuscript, where it twice appears.

Suppose that the square *a b*, having unknown sides, represents x^2 and apply to it a rectangular parallelogram of which the breadth is equal to one side of the square *a b* and the length is any quantity you please.[1] Then the numerical value of this rectangle is 21, which number accompanies the same x^2. Moreover this area or rectangle is called *b g*, of which one side is *g d* and the length of the two areas together is finally *h d*. And it is now evident that this length represents 10, since every quadrilateral having right angles (every square) gives for the product of one of its sides by unity one root. and if multiplied by two gives two roots of its area.

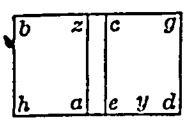


Fig. 8.[2] — From the Dresden manuscript.

Therefore since the problem was given, x^2 and 21 units equal 10 roots, it is evident that the length of the side *h d* is 10, for the side *h b* designates one root of x^2. Therefore bisect the side *h d* at the point *e*[3] so that the line *e h* is equal to the line *e d*. From the point *e* draw the perpendicular *e t*. This perpendicular equals *h a*. Add to the prolongation of the line *e t* a part *e c* equal to the amount by which it is less than *d e* and then *t c* will equal *t g*. Whence we arrive at the square *t l* which is the product of half of the roots multiplied by itself, that is the product, in this instance, of 5 and 5. Moreover we know that the area *b g* which we add to x^2 amounts to 21. Therefore we cut across the area *b g* with the line *t c*, which is one side of the area *t l*, and thus decrease the area *b g* by the amount of the area *b t*. Then we form

[1] Scheybl's text is incorrect here.

[2] If the lettering of this figure is made to conform to that of our text the demonstration will be seen to be not materially different; it is based more directly on Euclid II, 5. This proposition, following Heath, *The Thirteen Books of Euclid's Elements*, reads as follows: "If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half." This is one of the propositions of Euclid which connect very directly with the geometrical solution of the quadratic equation.

[3] The completed figure (Fig. 9) appears on p. 85. The lettering of Fig. 7 does not correspond to that of the completed figure.

the square *e n m c* upon the line *e c*, which is of the length by which the line *t e c* exceeds the line *a h*. Whence, since *t c* equals *l c*, being found in the square *t l*, and similarly since *e c* equals *m c*, these being equal dimensions of the square *e m*, and further equal lines being subtracted from equal lines, it is evident that *t e* is left equal to *l m*. This is to be noted.

Again it is evident that the line *g d* is equal to *a h*, since they represent in breadth equal dimensions of the area *h g*, and the line *a h* equals *h b* as they appear in one square. Also since the line *g l* is equal to *d e*, being found in the same square, and *d e* is equal to *h e*, each being the half of ten roots, therefore the line *d l*, the residuum of the line *g l*, is equal to *e b*, the residuum of the line *e h*. And so, as the line *t e*, by the above demonstration, is not unequal to the line *l m*, the area which is included by the lines *t e* and *e b* is equal to the area comprehended by the lines *l m* and *d l*. Therefore the area *t b* equals the area *m d*. The square *t l* equals 25. Therefore when we subtract from this same square *t l* the areas *d t* and *m d*, which are of course equal to the two areas *g e* and *t b*, containing 21, it is evident that we have left the square *n c*, which amounts to the difference between 25 and 21. This number is four, of which the root is two, and this gives the line *e c*. Moreover *e c* equals *d l*, since each represents the breadth of the area *d c*. Since *d l* equals *e b*, it is evident that when *e b*, which is two, is taken from *e h*, which is half of the roots, or five, three remains for the line *b h*. Therefore three is the root of the first x^2.

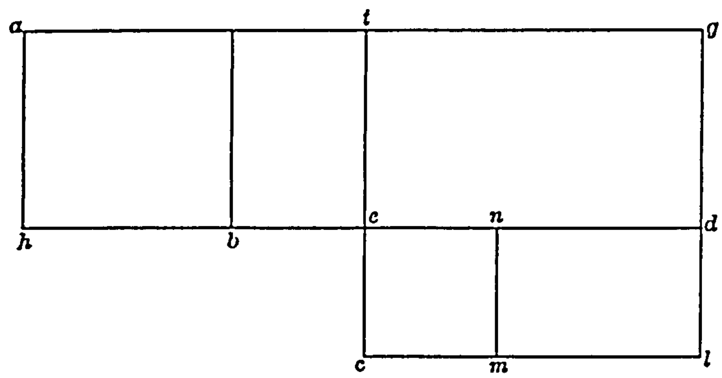


Fig. 9. — Completed figure. From the Columbia manuscript, where it appears twice.

On the contrary if we add the line *e c* to the line *e h*, representing half of the roots, we get 7 which is *n h*. And so the root of the square is greater than (the root of) the

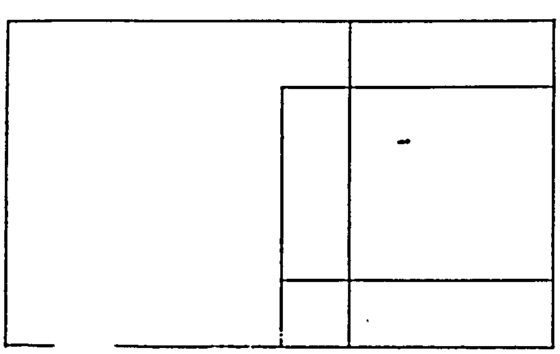


Fig. 10. — Unlettered figure. From the Columbia manuscript.

first square. Of course when you add 21 to it the sum is equal likewise to ten of its roots which we desired to demonstrate.[1]

[1] This paragraph is not found in the Libri version but appears in the Arabic as published by Rosen. The translation follows the Vienna version. The figure to be used in the geometrical demonstration to obtain by addition the second root of the given quadratic equation appears at the bottom of the preceding page. (Fig. 10.) The Boncompagni version (*loc. cit.* p. 35) varies by letting the middle point fall first within the side of the first square, and secondly without: Cum itaque dividitur per medium linea *b e* ad punctum *z*, cadet ergo inter puncta *g e* aut *b g*: sit hoc prius inter puncta *b g*

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