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Visual estimation of discrete quantities[1]

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Abstract: Visual estimation occurs when one is presented with a large group of objects for a short period of time and is asked to evaluate their number. Four strategies were expressed by third grade children in visual estimation situations: counting, grouping, comparison and global perception. After going through some visual estimation activities, some changes in children strategies were found.

Kurzreferat: Von visuellem Schätzen ist dann die Rede, wenn eine große Anzahl von Objekten für eine kurze Zeit gezeigt wird und die Anzahl der Objekte anzugeben ist. Schüler einer dritten Klasse wendeten vier verschiede Strategien beim visuellen Schätzen an: Zählen, Bündeln, Vergleichen und globales Wahrnehmen. Die Erfahrung mit mehreren Aktivitäten des visuellen Schätzens führte zu einigen Strategieänderungen bei den Kindern.

ZDM-Classification: C32, N22

1. Introduction

Visual estimation is undoubtedly an important component of visualization. It contributes not only to develop abilities of estimation as well as visual abilities, but it also helps to develop a basis for "feeling" quantities and numbers.

Visual estimation occurs when one is presented with a large group of objects (e.g. Fig. 1) for a short period of time, and is asked to evaluate their number. The ability to get as close as possible to the exact number is that kind of estimation where visual perception plays an important role. According to Bryant (1974), human beings are born with the ability of gathering information by *relative codes* only, and not by *absolute codes* meaning that children can say that one object is bigger (heavier, sweeter, etc.) than another object, but they cannot say how big it is. With experience, they learn to use *external frames of reference* and gradually they learn to deal with absolute information. In this sense, any previous "chunk of information", that children have, can act as a frame of reference (unit of measurement, absolute information about another object, etc.). Bryant's theory is that a young child's perception and memory of individual objects will be greatly influenced by their relations to their surrounding frames of reference.

The situation in which subjects are presented with a picture and asked "how many objects are there" is the basis of many psychological research studies. The main questions studied deal with the largest *exact* number of objects that can be identified and the subject's "reaction time" which is the time between the exposure and the subjects' answer. It was found that, when time was restricted, the largest number of objects one can exactly recall was 3-8 objects and that reaction time was a function of the number of objects. The arrangement of the objects was found to play an important role as well.

In the above studies, three strategies used by the subjects were identified: counting, estimation and the third has different names, such as "perceptual apprehension", "perceptual chunking", "numerical judgement" and "subitiz-

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ing" which means the immediate exact perception of the number of elements in a set shown for a short period of time (Folk et al., 1988). There is an argument among researchers about the existence of this last strategy, and those who say it exists, argue whether it can be learned.

2. The study

The study reported in this paper was performed as a part of the "Agam project" for visual education. This project is an example of an effort to interweave the development of a visual language with a process of developing visual thinking. This program is the vision of the artist Yaacov Agam that has become an educational reality through the work of a team of researchers and educators of the Science Teaching Department in the Weizmann Institute, Rehovot (Israel). The program was developed, tested and implemented with several groups of students beginning in nursery school with three-to-four year-olds and continuing with the same groups to the third grade. The development and implementation was accompanied by research and evaluation that showed that the "Agam children" can apply visual abilities and visual thinking in learning tasks more successfully than children in the control groups (Razel and Eylon 1990). Some of the program's thirty -six curriculum units introduce students to such basic visual concepts as the main geometric figures, directions, colors, and size relationships. These units make up a "visual alphabet" that forms the basis for more advanced concepts, such as symmetry, ratio and proportion and other concepts that serve as building blocks in scientific and mathematical thinking.

Unit No. 28 deals with "numerical intuition" which in this context was evidenced when a group of objects was shown briefly to students and they had to determine the number of objects without counting. Twelve third graders were interviewed twice: before the instructional unit and immediately afterwards.

2.1 The unit

The main objective of the numerical intuition unit was to develop children's visual numerical intuition. The unit included the following activities:

1) A set of 30 pictures was used. The number of objects (mainly dots) varied from 10 to 40 and the arrangement of the objects varied as well (Fig. 1. a,b,c). The children were presented with each picture for a short period of time and then asked to write down how many objects they had seen. As immediate feedback, the teacher showed them the exact number of objects, and the children wrote it down next to their answers.

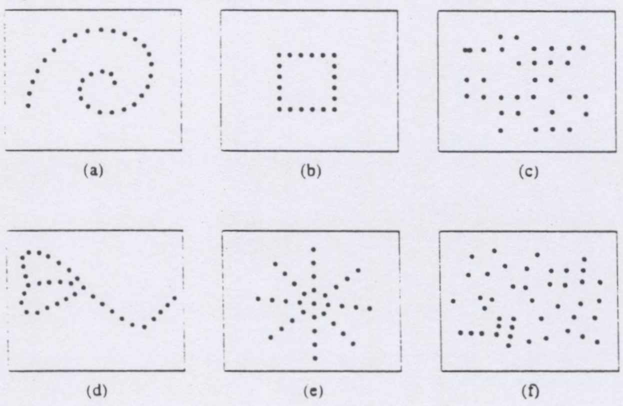

Fig 1

2) A short discussion was held on the idea that it is not a "failure" not to get the exact number. The idea of being "closer" or "less close" to the exact number was discussed.
3) Activities similar to (1) above were performed, but with concrete objects, such as plastic circles, plastic squares, pencils, which were spread randomly on a table; pictures in children books and objects from the classroom environment were also used.
4) Numerical intuition activities outside the classroom.

At no stage, during the numerical intuition unit , children discussed the strategies they used to evaluate the number of objects.

2.2 The goals of the study

1) To study children's visual estimation abilities.
2) To identify the kinds of strategies subjects used in visual estimation situations.
3) To investigate the influence of the unit on 1) and 2) above.
4) To investigate whether subjects are aware of how close their estimates were to the exact number of objects, or in other words to the idea of absolute and relative error. This last question will not be discussed here.

2.3 The interviews

In the first interview three pictures were presented, with 31, 20 and 32 dots respectively, (Fig. 1. a,b,c). A marble necklace with 189 marbles was also presented. In the second interview different dot pictures were presented with 33, 33, and 42 dots (Fig. 1. d,e,f). The dots arrangement of the pictures in the first interview were similar to those in the second interview. The same necklace was presented again. In both interviews dot pictures were shown for a short period of time that did not allow counting, while the necklace was shown for a longer period and the children were allowed to take it in their hands. In each situation, the children were first asked to evaluate the number of objects, and then to explain how they obtained the number.

3. Results

3.1 General findings

1) Children are not used to estimates. They prefer exact answers. Many of the children asked if they were supposed to count the objects in the picture. They were bothered by the fact that their answers would not be the exact ones. This is illustrated in the following exchange, (I – interviewer, C – child):

I: I'll show you a picture for a short time. Please tell me how many dots are in the picture (Fig. 1a).
C: (After the picture was shown) How can I tell?
I: You can tell about how many.
C: (After a long pause) About 37.
I: How did you get the number, what did you think?
C: (After a long pause) Let's say that I saw about a fourth. I counted how many objects are in a fourth and multiplied by 4. But this is not exact.

This student is a very bright one, and very good in math. During both interviews and during the numerical intuition unit he "suffered" a lot, being unable to give exact answers, as he usually does in math. He seemed to be very frustrated, especially in the second interview, being unable to calculate and give an exact answer, for example: "Thirty (Fig. 1f). I didn't do anything. At first I tried to count and to calculate, but I didn't succeed. I was confused and I was wrong all the time".

For many students it was difficult to accept answers close to the exact number, as good answers. This led them sometimes to say "I goofed" when their answer was only one or two away from the exact number. Only during the latest stages of learning, sentences such as: "I was close" were heard in the classroom.

This difficulty with estimates is typical for many other aspects of estimation.

2) Most of the answers were underestimates. For the dot pictures, 2/3 of the answers were underestimates, only 1/3 overestimates. (Exact answers were given in very few cases.) The phenomenon of underestimation was much more in evidence for the necklace. Here *all* answers were underestimates and were far below the correct 189. The average number of children's estimates was 65 in the first interview and 81 in the second.
3) As could be expected, children were closer to the exact number when the dots were arranged in some geometrical pattern (Fig 1b and 1e).
4) Overall, children were closer to the exact number in the second interview than in the first. The average relative error for the three pictures was 27% for the first interview and only 14% in the second. As to the necklace, although being far away in both interviews, children did better in the second.

3.2 Strategies of explanations

The following strategies were identified:

I) *Counting strategy* – children using this strategy counted as many objects as they could in the short period of time available. Then they added some more. For example: "38 (Fig. 1f). I counted up to 20. I counted as long as the time allowed". Or, "20 (Fig. 1a). I counted 2,4,6 and arrived to 6, and I still had many points".

It seems that children using this strategy, use counting which is a familiar and secure algorithm, as long as time allows. When they have to stop using it, they consider the number (quantity) they already obtained as a frame of reference to estimate the remaining quantity. This second step involves perceptual processes.

At this stage we don't have enough information about these processes.

II) *Grouping strategy* – children using this strategy mentally divided the objects into groups, most of the time with equal numbers of objects in each, and then multiplied this number by the number of groups. For example: "24 (Fig. 1b). It had 6 on each side, and 6x4 is 24". Or, "30 (Fig. 1c) The points are spread about 4 in each place. I circled groups of 4 in my head." It seems that children using this strategy used counting or perceptual abilities to evaluate the number of objects in each group, a number that was small. This is in agreement with the studies showing that subjects can perceptually evaluate the number of objects up to 8. The *size of the group* is now used as *external frame of reference.*

III) *Comparison strategy* – children using this strategy compared the number of objects to something they were familiar with. For example, "50 (Fig. 1c). It seems to me more than the two previous pictures". Or "20 (Fig. 1b). I relied on the previous picture, and took off a few dots". For the necklace, the comparison was more concrete: "50, I have at home a similar necklace".

Children using this strategy considered some *familiar absolute information* as *external frame of reference.*

IV) *Global perception strategy* – Children using this strategy did not use any other strategy or at least didn't express it. They just took a glance and gave their number. For example: "50 (Fig. 1f). I didn't count at all. It's according to the points. You can see it is 50". Or the following:

C: "25 (Fig. 1d). I simply looked and guessed".
I: "You just guessed?"
C: "No, I guessed according to what I saw. I looked and guessed according to what I saw".

It is difficult to analyze what lies behind this strategy. Is it that children just "have" this global perceptual ability, or do they use some other strategy unconsciously?

It seems that the strategy used depends both on individual differences between children and on the characteristics of the tasks. Some geometrical arrangement of the dots led children to use the grouping strategy more than any other strategy, and about 2/3 of the strategies used in figure 1b and 1e, were grouping strategies. On the other hand, there were children who used the counting strategy for the necklace, even though it seemed to be very inefficient in that situation.

Most of the children were not systematic in their responses and did not use the same strategy in all three pictures, but some of them did. For example, three of the children used the counting strategy for all the pictures in the first interview.

3.3 Change in strategy

The most dramatical change in the strategies, when comparing the two interviews, was the decrease of the counting strategy from 42% in the first interview, to 19% in the second, and the increase of the global perception strategy from 16% in the first interview to 34% in the second. (The percentage is calculated out of 36 explanations given for the three pictures by 12 children.)

It seems that the experience with this kind of situation during the numerical intuition unit, directed some of the children to realize that the counting strategy is not a very efficient one, so they abandoned it in favor of a more efficient strategy like global perception, which was used in 1/3 of the answers in the second interview. The following is an illustration for this change. A child explained in the first interview: "I counted 2, 4, 6 up to 6". Or, "I started to count 2, 4, 6, and I reached the 8" (Fig. 1a and 1c). In the second interview the explanation of this child was quite different: "That's what I saw, I didn't start counting, I looked". Or, "There are many dots, it looks to me that there are 40".

Two children didn't change their counting strategies at all. They constantly used this strategy in the first interview, and continued to use it in the second as well.

The use of the two other strategies didn't change much from the first interview to the second. The grouping strategy increased from 31% in the first interview to 36% in the second, and the comparison strategy didn't change at all; it was 11% in both interviews

4. Concluding remarks

- In the above study we found that children's estimates in the second interview were better than in the first interview. It seems that by gathering experience – going through the numerical intuition unit – children can "educate their eye". In addition we found that children

had changed their strategies, abandoning the counting strategy in favour of the global perception strategy.
- The perceptual element is believed to act underneath each strategy, where the difference is in the "size of the quantity" that was globally perceived. In the grouping strategy, for example, the children perceptually evaluated the group size and/or the number of groups.
- In each of the first three strategies, frames of reference were used: In the first one the "counted quantity" was used as the frame of reference in estimating the remaining dots. In the grouping strategy the size of the group was used as the frame of reference, and in the comparison strategy some external but familiar information was used as a frame of reference.
- While in the grouping and comparison strategies children chose to use frames of reference, in the counting strategy, the strategy itself, because of the shortage of time, forced them to rely on some frames of reference.
- We do not really know what lies behind the global perception strategy: Is it a different strategy or is it an acceleration of one or more of the other strategies, which is done unconsciously via rapid inferences (Lesh and Mierkiewicz 1978)?
- It is worth while to note that some children, in the explanation of their strategies, related to the mental images of the dot pictures that they had created in their minds rather than to the real pictures. This was found to be independent of the strategy used.
- While going through the numerical intuition activities, children gathered experiences in visual estimation on their own, meaning that there was no guided learning concerning the strategies to be used, (strategies were not discussed at all in the unit activities). It would be interesting to investigate the influence of direct discussion with children about the strategies they use, on the above change in strategy.

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Vorschau auf Analysenthemen der nächsten Hefte

Für die Analysen der Jahrgänge 25 (1993) und 26 (1994) sind folgende Themen geplant:
- Mathematik und Allgemeinbildung
- Chaos und Fraktale
- Frau und Mathematik
- Visualisierung
- Umwelterziehung im Mathematikunterricht

Vorschläge für Beiträge zu o.g. Themen erbitten wir an die Schriftleitung.

Outlook on future topics

The following subjects are intended for the analysis sections of Vol. 25 (1993) and Vol. 26 (1994):
- Mathematics and general education
- Chaos und fractals
- Women and mathematics
- Visualization
- Environmental education in the mathematics classroom

Suggestions for contributions to these subjects should be addressed to the editor.