Chapter 5: Space and Shape

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ABSTRACT

Three perspectives about the possible roles of shape and space are discussed:

1) Interacting with real shapes in space,
2) Shape and space as fundamental ingredients for constructing a theory, and
3) Shapes or visual representations as a means for better understanding concepts, processes, and phenomena in different areas of mathematics and science.

Shapes are considered as dynamic entities where their ability to change is one of their main characteristics. The meaning of shapes is also changing, both between and within, each of the above perspectives.

Student's ability to visualize real objects, mathematical concepts, processes, and phenomena is considered now as a mathematical activity like computing or symbolizing. Nevertheless, unlike numerical or algebraic education, visual education is often a neglected area in curricula, in particular in relation with the first perspective. This is the reason why the authors choose to describe three projects that invested efforts in a systematic development of visual education.

1. BACKGROUND: ABOUT SHAPE AND SPACE

What do mathematicians and mathematics educators mean when they speak about shape and space, or their role in education, or even more explicitly in mathematics education? This is not a very easy question and possible answers are not clear-cut.

Marjorie Senechal (1990), in her triggering chapter about shapes relates to this question, claims that shapes are patterns and that the concept of shape, like many other concepts, is an undefinable term. On the other hand she said: 'We assume we know what shapes are more or less: we know one when we see one, whether we see it with our eyes or in our imaginations' (p. 140). If we have difficulties with defining what shape is, we definitely have difficulties to describe what we mean, when we talk about space – The world around

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us? The world of shapes and relations between them? And what does this mean? What is the kind of thinking we have to develop in order to interact successfully with this world, and what does it mean to interact successfully?

If one tries to come closer to these kind of questions, and to get some ideas of possible answers, one has to discuss the following three perspectives:

- Interaction with real shapes in space (see 1.1)
- Shape and space as fundamental ingredients in the construction of a theory (see 1.2)
- Shapes or visual representations as means for better understanding concepts, processes, and phenomena in different areas of mathematics and science. (see 1.3)

As can be inferred from the following, the boundaries between the three are quite vague.

1.1 First perspective: Interacting with real shapes in space

Interacting with real shapes involves understanding the visual world around us (mostly what can be seen), its description, encoding and decoding of visual information that flows plentifully in this world. It also means interpretation of visual information. And more important, integrating all the above to handle, to cause and to understand changes on the shapes that populate our space. This is the space in which we have to deal with realistic problems typical to many areas; from simple everyday problems to problems in architecture, art, engineering etc.

According to Senechal (1990) meaningful interaction with real shapes in our space has three main goals: 'To discover similarities and differences among objects, to analyse the components of form, and to recognize shapes in different representations' (p. 140). She discusses and suggests a strand of topics related to shapes, through all school years, guided by three main tools: identification and classification of shapes, analysis of forms and representations and visualization of shapes.

In addition to Senechal's claims, we would like also to emphasize *dynamic aspects,* such as the relative position of several shapes to each other, the relative position of an observer and the things that the observer looks at, and the processes of changing shapes.

1.2 Second perspective: Shape and space as fundamental ingredients for constructing a theory

The traditional geometry, which was codified by Euclid more than 2000 years ago, had started from what can be seen with the eyes. Space and the shapes provide the environment in which the learner can get the feeling for a mathematical theory (Freudenthal 1973). This kind of environment, at a more advanced stage, acquires a broader and more abstract aspect, without the necessity of a real environment as a basis. (e.g. multi-dimensional Euclidean geometry or non-Euclidean geometries). But, even in the most abstract case we still deal with shapes and some sort of space, even when their visual representations cannot be seen with the eyes. What is left are representations in the minds' eyes (mental images), or in other words the theoretical representations.

1.3 Third perspective: Shapes or visual representations, as means for better understanding concepts, processes, and phenomena in different areas of mathematics, science and related subjects.

This perspective was discussed intensively by educators in the last decade (Eisenberg & Dreyfus 1991; Clements & Battista 1992). In the introduction to their book: *'Visualization in teaching and learning mathematics'*, Zimmerman & Cunningham (1991), even speak about 'the visualization renaissance' (p. 1). (We understand visualization as the transfer of objects, concepts, phenomenon, processes and their representations to some kind of visual representations and vice versa. This includes also the transfer from one type of visual representations to another.)

Researchers usually speak of one main reason for this renaissance. In our modern life the presentation of phenomena has changed from tables and formulae heavy with numbers and symbols, to a dynamic visual presentation on the computer monitor. The computer visualization becomes a scientific and mathematical tool. In order to understand, analyse and predict, we will have to engage in some visual thinking.

The second reason for the renaissance is expressed by Zimmerman & Cunningham (1991), saying that currently there are changes in the view of the nature of mathematics, whereby mathematics is seen as an on-going 'search for patterns' (Steen 1988, 1990), and this metaphor is surely a visual one. Therefore: 'It is natural to try to find the most effective ways to visualize these patterns and to learn to use visualization creatively as a tool for understanding' (p. 3).

1.4 The changeable nature of shapes and how it relates to visual thinking

In the following we will relate to shapes not only as static entities but rather as dynamic entities where their ability to change (not only in meaning) is one of their main characteristics. Shapes have different meaning and their role is to keep changing between and within the above perspectives. In the first perspective, where we relate to shape and space for its own sake, we are interested mostly in real objects which populate the real world, their shapes, and relations between them. There are many levels of abstraction concerning the meaning of shape in this perspective – for example, when we classify objects according to some chosen properties we eventually come to the essence of a shape as a representative of a whole class of objects. The abilities to perform different kind of classifications according to different kinds of characterizations (e.g. similarity and self similarity, combinatorial etc., (Senechal 1990)), to analyse different objects into their primitive shapes and above all to interpret and describe information from our real world are only some of the ingredients of visual thinking needed here.

The second perspective stresses a dual nature of shapes (here mainly geometrical figures); as Laborde (1993) expressed it: 'Figures can be viewed as playing the role of reality with respect to theory as well as playing the role of model for a geometrical theory' (p. 49).

The word 'figure' itself is ambiguous, because it can refer either to a geometrical object (i.e. conceptual, or 'idealis' in a platonic sense) or to a graphical representation (i.e. material) of such an object. This ambiguity is a well known source of difficulty for younger students because they do not understand that the objects referred to by their teacher are not the drawings (diagrams) which they can see in their textbooks, or on the blackboard, or that they realize themselves.

In this case the 'figures' (diagrams) are in fact tools for solving problems or to construct a geometrical theory (Euclidean geometry), and the process can be schematized by:

Diagram → Figure → Geometrical Theory.

But, in a dual way, a geometrical theory may be made more accessible by using a 'model' which can be visualized as for instance the Poincaré disc model with regard to the hyperbolic plane.

In this case the 'reality' (i.e. diagrams made on a sheet of paper) is not the starting point of the cognitive process, but of a process that is the reverse of the previous one:

Geometrical Theory → Figure (Model) → Diagram.

In the third perspective shapes are considered as the visual representations of mathematical or scientific entities (concepts, processes, phenomena).

Here in addition to the ingredients of visual thinking mentioned above, we have to develop the ability to interpret, understand, and create, the mutual relationships and analogies between the visual representations and other kinds of representations (Kaput 1992).

The various perspectives of shape and space on the one hand, and the interplay between concrete and abstraction on the other hand, are what make visual thinking both very rich but quite complicated; they are at the same time intuitive, global and analytical.

2. WHY VISUAL EDUCATION?

Visual education is needed for effective and correct interaction with shapes, relationships between them, transformations on shapes, relationships between shapes and other entities etc. This is true for the three perspectives mentioned above because whenever students have to relate to shapes and space, whether in their eyes or in their minds' eyes they have to apply some sort of visual thinking. Visual ways of thinking and reasoning may be acquired through a well planned visual education.

There is now a consensus that in the student's search for mathematical patterns and relationships within the various perspective mentioned above, visualizing is to be considered a mathematical action like computing, or symbolizing, (NCTM 1989; Senechal 1990). Nevertheless visual education is often a neglected area in curricula. This is especially true if we speak about the first perspective: interacting with the real world.

The reality of mathematics education in many cases only touches visualization and visual shapes indirectly:

i) This happens in the traditional Euclid geometry course, which is still the most important school geometry course in many parts of the world. In such a course, where deductive reasoning is the main goal and shapes are the theoretical objects, knowing shapes and visual reasoning are considered, on one hand, as the intuitive basis for the development of higher levels of reasoning (See the Van Hieles' theory in Van Hiele 1987, Hoffer 1983, Crowley 1987). On the other hand, visual reasoning is considered as a low level reasoning that can even distort students' logical reasoning (Schoenfeld 1986). For example, in France, in the '1970's, this way of thinking about geometry was brought to its extreme consequences, when diagrams were positively prohibited from the teaching of geometry.

ii) Lately, with the boom in changes in the views about mathematics education, where visualization in mathematics (the third perspective) becomes a popular subject, the development of visualization and visual thinking is expressed in the learning of the visual representations of mathematical concepts. Here also the main goal of teaching and learning is not the visual shapes for their own sake. In addition, for many years teaching multi-representational processes in mathematics meant a more or less systematic way of looking for dynamic processes in the analysis of algebraic and numeric representations. There was much less of a systematic approach to such processes in visual representations.

It seems that in both cases there is a hidden naive assumption that somehow students do have visual thinking abilities and that they apply visual reasoning when they have to. We can realize how absurd this situation of visual education in the curricula is when we make an analogy between this area and other similar areas like verbal education or symbol education or the familiarity with the world of numbers and operations in education. It is hard to explain how it is possible that visualization, which is declared to be in its renaissance, does not receive a similar systematic attention in the curricula as other subjects do?

In addition, there are a few other very crucial reasons for investing systematic efforts in the development of visual thinking along preschool and school years. In the following we will mention a few important aspects.

Cognitive aspects:

- visualization is an essential part of human intelligence;
- visual development does not occur through a linear approach;
- phenomenological approach to learning mathematics can give the student a better understanding of space and shape (starting in problem situation, searching for patterns, rich contexts instead of poor ones, the role of re-invention).

Another aspect is a social one: Emancipation of society, which results in demands for new ways of teaching and the fact that more people get general education and the traditional (Euclidean) approach to learning and teaching geometry, does not fit all of them. We assume that alternative ways of learning geometry will stress visual thinking and may relay on the dynamic graphical powers of technological tools.

A third aspect arises out of changes in the view on the nature of mathematics. We are moving from a view that mathematics consists of a logical structure that we have to follow (or discover) towards a view in which mathematics is (also) a process of conjecturing and justification or refutation. Here again the experimental environment for conjecturing, that involves the use of visual objects should play an important role.

Finally there has been for many years a gradual change in view on the nature of mathematics education, in particular on the kind of mathematical activities. The main idea is to have the students actively involved in the learning situation, which they create and accept as a problem situation within their reality.

3. SOME EXAMPLES OF TEACHING 'SHAPE AND SPACE'

3.1 Introduction

In the following, we intend to describe three examples, in which we see significant and important changes in visual education.

We are aware that there are many efforts in the world on developing visual education. We think, however, that there are not so many on the first perspective: visual education for interaction with real space. In the following three examples the focus is mainly on this perspective, both for its own sake and as a prerequisite for the other two.

3.2 The Agam project as an example for a 'culture' of visual education

3.2.1 Goals and assumptions (beliefs)

The central goal of the Agam program is to develop young children's abilities to perceive, think, and create – in a wide variety of domains, by using a visual language. This goal is based on three beliefs:

a) *The value of visual thinking;* As was discussed in section 1, it seems that visual thinking is *the* thinking of the very near future. In order to understand, analyse and predict and communicate, we will have to engage in some visual thinking. Visually communication, and visual thinking, as in other domains, depends on the use of certain linguistic elements: a visual language.

b) *The value of systematic training;* As was mentioned above formal education stresses mostly verbal and symbolic elements where the training of visual elements has been traditionally neglected. But, as in the verbal or symbolic case, visual language and visual concepts and processes be systematically trained and acquired.

c) *The value of early training;* Teaching young children to think visually can significantly contribute to their cognitive development in many domains, including mathematics and science. In this regard, the program's content can be viewed as one of the basics.

3.2.2 Description of the Agam program

The Agam program is an example of an effort to interweave the development of a visual language with a process of developing visual thinking. This project is the vision of the artist Yaacov Agam (Agam 1984), that has become an educational reality through the work of a team of researchers and educators (Razel & Eylon 1986; 1990; Eylon & Rosenfeld 1990; Hershkowitz & Markovits

1992; Markovits & Hershkowitz 1996). The program was developed, tested and implemented with a number of groups of children, beginning in nursery school – 3 to 4 years old, and continued with the same groups up to the third grade. The development and implementation was accompanied by research and evaluation.

The program was planned to contain 36 units (see Table 1) most of them already developed and tested. Some units introduce children to basic visual concepts, such as *some main geometrical figures* and *concepts* (e.g. units 1, 2, 11, and 17 in Table 1) *directions* (e.g. units 6, 7, 8, 9 and 10), *colors* (e.g. units 21, 22, 23, 24 and 25) *size relationships* (e.g. units 16) and certain visual skills such as *variations of forms* or *visual composition*. These units make up a visual alphabet that form the basis for more advanced units on concepts such *as symmetry, ratio and proportion, numerical intuition*, and *dimensions*. These advanced units can be considered as some of the building blocks in scientific and mathematical thinking.

1. Circle	19. Typical Forms
2. Square	20. Proportions
3. Patterns	21. Red
4. Circle & Square	22. Yellow
5. Flash Identification	23. Blue
6. Horizontal	24. Secondary Colours
7. Vertical	25. White, Black & Gray
8. Horizontal & Vertical	26. Trajectory
9. Oblique	27. From Eye to Hand
10. Horizontal, Vertical & Oblique	28. Numerical Intuition
11. Triangle	29. Composition
12. Circle, Square & Triangle	30. First Dimension
13. Variation of Forms	31. Second Dimension
14. Symmetry	32. Third Dimension
15. Curved Line	33. Fourth Dimension
16. Large, Medium & Small	34. Letters
17. Angles	35. Visual Grammar
18. Point	36. Creativity

Table 1. The Units in the Agam Project

According to Agam (1984), a primary objective of this method is 'to educate the eye'. This aspect is expressed in the visual meta-processes that the program develops intentionally within each visual concept (unit), and between the units. Razel & Eylon (1990) discuss the following processes:

a) Analysis of forms into their basic elements and synthesis of simple elements to complex visual forms.
b) Accuracy of visual encoding and decoding and immediate and direct visual perception.
c) Perceptual flexibility, which is expressed:
   i) in the ability to move between an analytic and synthetic mode of perception discussed above.
   ii) in the ability to perceive the invariants of shapes under perceptual changes (e.g. the red big square in upright position is still a square when rotated or shown in any other position).
   iii) in ability to choose the right level of detail in visual perception and make use of it in a given situation. For example, in the numerical intuition unit, which is one of the latest units (Markovits & Hershkowitz 1996), the third-grade child is presented with flash cards bearing various numbers of dots for a short period of time, and asked to evaluate the dots' number (Figure 1).

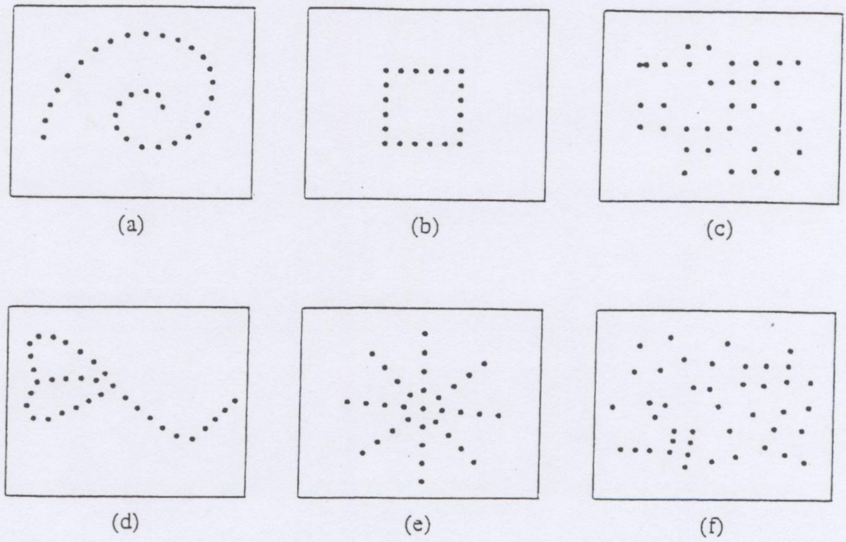


Figure 1. Some patterns in the numerical intuition unit.

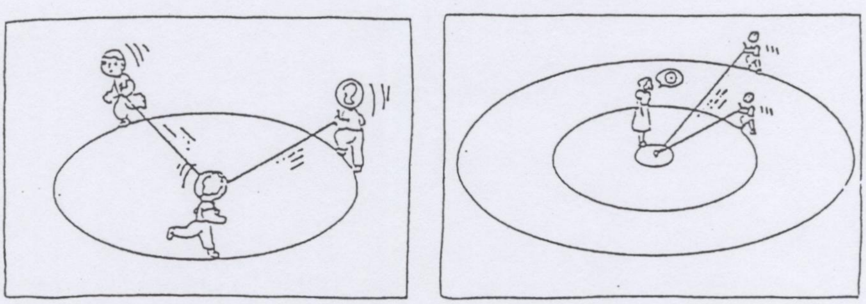
One of the strategies used by children is the *group strategy*, in which the child mentally divided the dots into groups, usually equal in number, which he or she then multiplied by the total number of groups. Perception was the basis for evaluating the number of objects in the group as well as the number of groups, and then it was used to overcome the difficulty of evaluating the total number of objects.

In the following we describe few examples by which we try to demonstrate some main features of the project

3.2.3 The Pattern Example (Hershkowitz & Markovits, 1992)

The example of the *pattern* unit demonstrates how visual language and visual thinking are developed by learning in the Agam program:

- The first unit in the program is *circle*, the second is *square* (see Table 1). The 3- to 4-year-old child becomes acquainted with these concepts by methods relying almost exclusively on non-verbal experiences: for example an activity in which children each take one end of a rope (the ropes being first of equal length (Figure 2a) and after that of different length (Figure 2b), whose other end is fixed to some point (the centre), and by walking on the created circles demonstrate visually and physically the circle properties (Figure 2).



(a) (b)

Figure 2. Students use ropes to make circles.

From circles and squares as basic linguistic elements (visual letter), and a combinations of a few squares or a few circles that can be considered as visual words, patterns (visual sentences) can be created (*patterns* is the third unit in the program). Pattern, in the sense used here, is a visual periodic series whose elements, at this stage, are circles and squares of different sizes, colors, and position, with changing intervals between them (Figure 3). The following excerpts are taken from the introduction to the teacher activities book of this unit, and they shed some light on the rationale of this unit and on the structure of the visual language as well:

> The patterns in this unit do not have beginning nor end, as if we look on the pattern through a window, where the window size and its position on the pattern cuts in an arbitrary way the seen part of the pattern...
>
> In the units square and circle, the child had learned some visual elements which can be considered as 'visual letters' in the visual alphabet. In addition the child had learned to create 'words' in this language where each of them consists on few squares or few circles combined together according to some 'combination law' (intersect, touch, include, on top of, under, etc.). In this unit the child learn to create 'sentences in the visual language' by using a 'grammar law' which determines a simple repetition on one 'word' (the period).
>
> (Agam 1985).

Periodic series are the basis of many mathematical and scientific concepts, such as certain functions, waves, movements. This unit tackles the concept visually in a way which is meaningful to the young child. When children identify patterns provided by the teacher, books, or the classroom environment, or when they memorize and store various patterns and recall them, they internalize the pattern concept and realize that it is the same irrespective of the changes in the periodic themes that create different patterns (Figure 3).

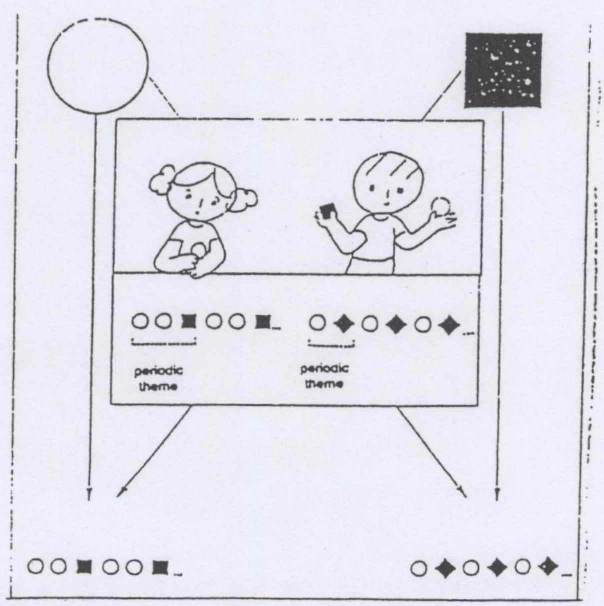
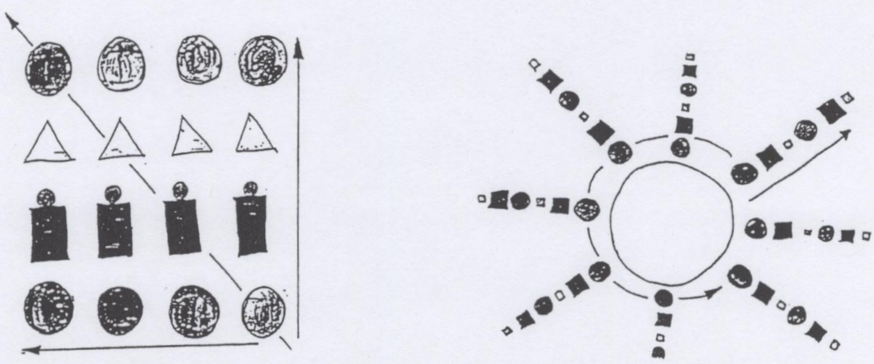


Figure 3. Patterns.

When children create patterns, they are in a problem-solving situation, with high level thinking. They have to analyse the main characteristics of pattern

– the building blocks that are to be used in its creation, and choose those that they would like to have in their own special pattern. Finally they have to synthesize all of the above in the production of their own pattern. In the patterns unit all the planned activities deal with linear patterns (Figure 3), but children's creativity is unbounded – for example, the sun (Figure 4a) in which there is the periodic series with a ray as the periodic theme and each ray itself is a linear pattern. The matrix (Figure 4b.) is also a multidimensional pattern: each row, each column and the two main diagonals exhibit a linear pattern and the whole matrix is a pattern with the column as the periodic theme.

Pedagogically, the program emphasizes a systematic and even quite structured approach between the different units and within each of them.



(a) (b)

Figure 4. Multidimensional periodic patterns created by 4 year-old students.

This approach is activity-oriented with many manipulative actions done in small groups of children. These actions suggest multiple representations of the same concept. In the unit on patterns children are given different auditory codes for circle and square; they are than asked to create a visual pattern by following an auditory dictation of these codes. In other activities, within the same unit, children represent the concept with different materials – for example, transparencies, plastic forms, construction paper, elements in the child's environment (a row of windows).

The teaching language in this program tries also to be visual as much as the effectiveness of learning allowed. At any case it includes only minimal verbal interventions which are mostly labelling or instructions and the verbal description of the concept or its properties.

3.2.4 The Ratio and Proportion example (Hershkowitz & Markovits, 1992)

This unit is one of the latest units in the program in which third graders who have completed many units of the program come closer to this higher level mathematical concept.

The following is a brief description of the sequence of activities that are the basis of the unit on Ratio and Proportion.

a) Sticks of various lengths are used, with a 'short one' as a unit ('unit stick'). Children count how many times 'the unit' can be put along two different sticks and draw conclusions about the *ratio* between the lengths of these sticks.
b) Repetition of the same activity with sticks of different lengths leads the children to discover that a ratio exists between certain sticks, regardless of their particular lengths.
c) Children are asked to find different pairs of sticks related in the same way. In fact they discover *proportion.*
d) Similar activities with towers of blocks, and the heights of a coloured liquid in glasses. The towers demonstrate that two ratios can be the same even if they are expressed by different numbers, e. g., 1/3 = 2/6 (Figure 5a). This is a visual basis for equality of fractions.

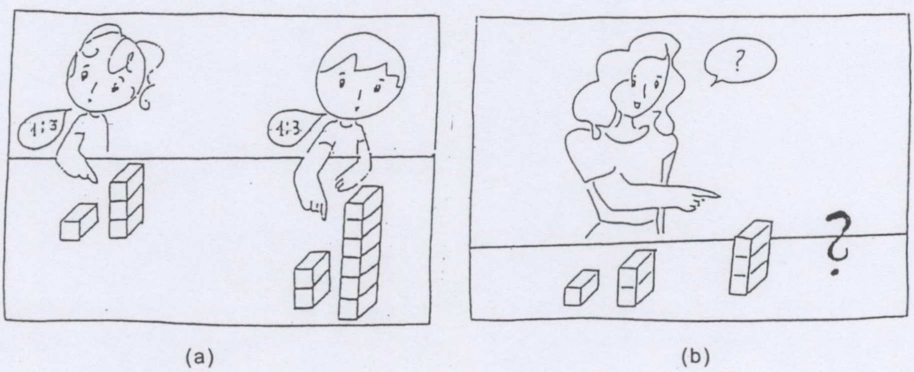


Figure 5. Relation and proportion demonstrated in block towers.

e) Similar activities with different size angles. (The *angle* unit comes earlier in the program). Children reinforce their intuition that the same ratios and proportions exist for different measures.
f) Children discover by measuring with sticks or paper strips that the ratio of the length of the sides of similar shapes is constant.
g) Children discover that the ratio between the lengths of two vertical sticks is equal to the ratio between their shadows (proportion).

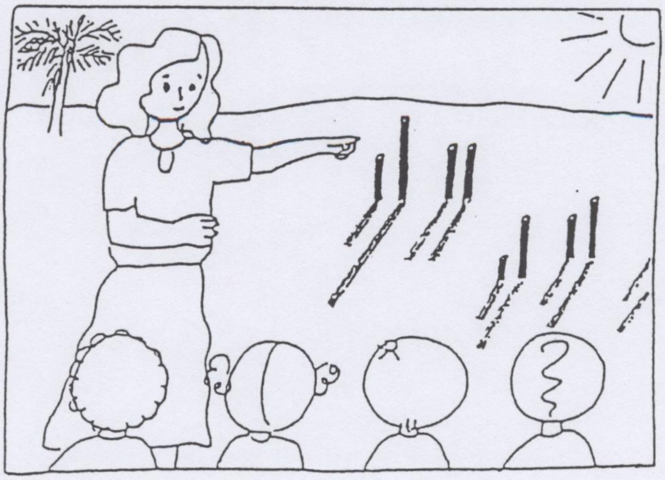


Figure 6. Students discover the proportion between the lengths of sticks and the lengths of their shadows.

h) The more advanced activities are those in which three values in a proportion are given and the child has to find the fourth one. For example, a pair of towers and a single tower are given: the child 'calculates' the height of the fourth tower which completes the proportion and then checks it by building the tower (Figure 5b), or the child 'calculates' the height of a vertical stick given the height of a second stick and the length of the two shadows (Figure 6).

From the classroom: As is to be expected, the level of learning varied between children and according to the type of task they had to deal with. The following are some examples of our observations of the children's learning.

- At the beginning (activities a, b and c), children leaned very much on the 'unit sticks'; for example, after they measured the lengths of two sticks in order to find the ratio, they would not agree to remove the 'unit sticks', even though they had already found the ratio. With time and accumulation of experiences most of them no longer needed it.
- In type c activities many children were satisfied if they found a single pair of sticks with a given ratio. With time, most of them succeeded in finding as many different pairs as one could obtain from the variety of sticks in their disposal.
- Some children used trial-and-error strategies in finding a pair of sticks in a given ratio, whereas others planned their steps very carefully. For example, when they had to find a pair in the ratio 2 to 3, they first took 5 'unit sticks', put 2 in a line and 3 in another line and only then found suitable sticks.

- When presented with 3 sticks and asked to produce a drawing of sticks in the same ratios, some children explained their drawing qualitatively as big, medium and small, while others succeeded in seeing and even expressing the ratios quantitatively.
- By the end of the unit, all the children were able to create different pairs of sticks, towers of blocks, etc. for a given ratio (proportion), but only few were able to successfully deal with activities in which 3 values in a proportion were given and the forth one had to be found.

3.2.5 Research goals and methods

The research goals were to:

- evaluate the program's implementation,
- assess the cognitive outcomes,
- assess the affective outcomes,
- explore cognitive processes of processing and documenting visual information.
- explore processes of visual thinking.

To achieve these goals, the staff adopted a research design which integrated quantitative and qualitative methods, which were viewed as complementary in nature:

a) quantitative methods focused on the use of cognitive tests in an experimental versus comparison group design with pairs of preschool classes (Eylon & Rosenfeld 1990). The children in the experimental schools studied and did group work in the Agam program bout three times a week. 20 – 30 minutes each time.
b) qualitative methods focused on the use of naturalistic observations, interviews, and case studies of specific children.

3.2.6 Research findings

A quantitative research was done after about one third of the program (Razel & Eylon 1990; Eylon and Rosenfeld 1990). Results showed that exposure of children to this part the Agam Program increased the general intelligence of the trained children in comparison with non-trained ones. Strong positive effect was also found on general school readiness, expressed in writing, geometry and logical thinking. The effects of the program were similar for middle-class as for lower-class children, where the visual performance of the lower-class children far exceeded those of the untrained middle-class children.

In addition the Agam children demonstrated a significantly greater ability to identify visual concepts in complex contexts, a better understanding of these concepts and a better application of them in complex visual settings.

The children also performed better on most visual memory tasks.

By qualitative research, team members tried to become close to cognitive processes of acquiring mathematical concepts by visual approaches – for example, the concept of ratio and proportion discussed above – and to the strategies in which children document, interpret and explain a certain visual information processes in which they are involved – for example, visual estimation of discrete quantities (Hershkowitz & Markovits 1992; Markovits & Hershkowitz 1996).

It was found that visual didactic means, like different pairs of sticks with a same ratio in the proportion case, or the visual relation between quantities of objects to be estimated in the visual estimation case, serve as a visual representation on which the child may construct his conceptualization. In these situations concepts like ratio and proportion and processes like proportional reasoning (in the judgment of visual estimation processes) are acquired quite differently and may be easily.

The program had especially dramatic effects on certain children in terms of their cognitive development and self-image. Children who tended to be introverted or nonverbal became very involved in the activities. As a consequence the program was also implemented with groups of children in special education. The special educators considered the Agam program as one of the most relevant treatment for that kind of population.

3.3 Space and shape and reality

Starting in the late 1960's and early 1970's, a kind of mathematics education has been developed in several countries all over the world. In the Netherlands this became known under the name *realistic mathematics education* (Broekman & Van Dormolen 1989, Feijs & Lange 1995; Freudenthal 1991; Gravemeijer 1994, 1995, Lange 1982, 1984, 1987; Schoemaker et al. 1981; Streefland 1991; Streefland 1993; Van Dormolen 1993). The term 'realistic' refers to reality, not to real life situations. Reality is in this view a subjective concept: it is the total of experiences and imagination of a person. In this view, a fairy tale can be realistic for someone, while for the same person many situations from daily life are not realistic at all.

Ideas behind this movement came from a concern that:

- mathematics education takes place in situations that student recognize and with which they can identify,
- mathematics education is focused on being engaged actively and mathematically justified with solving problems,
- mathematics is useful for other subjects at school and outside the school,

- students are in a position in which they get a good picture of their own ability to learn and to apply mathematics and work together with others according to these abilities,
- mathematics education prepares for education in the future (Van Dormolen, 1982).

Of course, these few sentences are far from being sufficient as a description of the principles of realistic mathematics education. Yet we shall not elaborate on the subject in general, but some remarks need to be made in order to describe clearly what came out of it in the case of changing and building up a curriculum for geometry, in which learning how to cope with space and shape is a basic part.

As a result of the principles above, realistic mathematics considers mathematics very much as an activity in which students and teacher work together and re-invent both mathematical ideas, as well as mathematical methods. Reflection on one's own work is, between others, an important activity.

Another central activity is mathematisation: to (re-)invent mathematical ideas and tools, to (re-)discover mathematical properties. Here we choose to use the terms *horizontal mathematisation* and *vertical mathematisation*, used by the realistic mathematics team (Treffers & Goffree 1985).

Horizontal mathematisation has to do with establishing a relation between non-mathematical situations and mathematical ideas. (Metaphorically this is like building a bridge among the two). Vertical mathematisation is an activity in which mathematical elements are put together, structured, organized, developed, etc. into other elements, often in a more abstract or formal form than the originals. (*Horizontal mathematisation* is sometimes called *modelling*. We choose to use the term *horizontal mathematisation* in this chapter because modelling can also happen in *vertical mathematisation*).

In this section we want to elaborate on elements of the curriculum, that are related to space and shape. Therefore we shall not describe the full curriculum. We shall follow a more categorized order and because of that it shall not always be clear in what grade a certain activity takes place. In fact a particular activity might take place in several, if not all grades, on different levels, of course. The headings of the subsequent sub-sections describe the categories.

3.3.1 Experiencing space

In the beginning we wrote about some ideas behind realistic mathematics. In following up these ideas, geometry education starts with orientation in real space, that is: the space of which students themselves are part of. Therefore experiencing space is about the relative position of objects in space and the relative position of the objects and the position of an observer of these objects. The student as the observer (or one who identifies with the observer) describes this relative position. There are several possibilities. For each possibility we shall first give some examples and afterwards discuss them in more general terms.

3.3.1.1 What I see

The following are three learning situations:

Objects on the table

Some objects are put on the table and the students are given three photographs of these objects. Each photograph is taken from another direction, but all with the lens on the same height as the table. Students are asked to decide where the photographer stood when each of these pictures were taken.

The singer

A picture is given of a television studio (Figure 7). One sees a singer and four television cameras. Camera A is in front of the singer, camera B looks from the right, camera C from the back and camera D from the left.

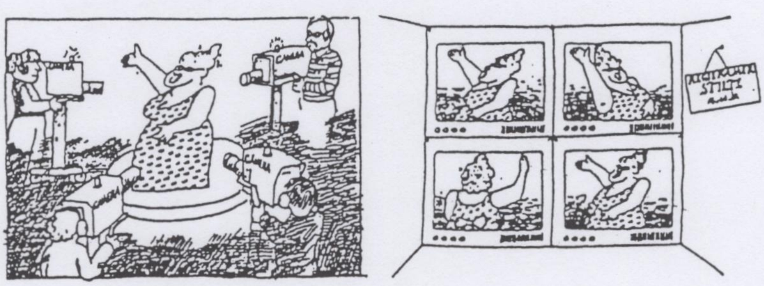


Figure 7. Singer.

There is also a picture of the four monitors in the director's room. Students are asked to play the director and as such they have to know which monitor belongs to which camera.

The viaduct

The students are given a photograph on which we see a viaduct and some persons standing on top of the viaduct. The photograph is taken from the road on one side of the viaduct. We see cars approaching, one of them a big, high van. Some cars are just passing under the viaduct and some are already on our side.

Students are asked questions like: If you were the photographer, would you be able to predict whether the van can pass under the viaduct?

In the first situation (objects on the table) the student is the observer. In the other two (singer and viaduct) the student can directly identify with the observer. The difference between the first situation and the second is that in the first case the students can walk around the table, while in the second they have to imagine the situation. If, however, this is too difficult at the moment, they can use blocks or other objects to put on the table. These objects play the role of the objects in the picture. Now the students can, as in the first situation, walk around the table and look at the objects in reality.

In the third situation, when students are asked to draw a picture and to explain what they saw, they most often draw a picture that is identical with the photograph, while a view from aside would have been more appropriate. The student has to do some transformation and to interpret what the observer saw and than to go back to the reality by predicting some results.

There is a direct relation between what the student sees and what the observer would see. This is because the situation fits in the student's own experiences. Such kind of problem situation are appropriate for the very beginning of learning about space and shape in geometry.

3.3.1.2 How I see

In almost the same situation as in the first example, the students are given three photographs, but this time they get the objects in hands and are asked to put them on the table in their relative position. They are also asked to draw a map (view from above) of the table with the objects. Then they are asked to indicate on the map where the observer stands in each case of the three pictures and other questions like: Draw what someone would see from one side. Where on that drawing would she or he be in order to see the photographer?

Again, as before, students who have great difficulties with problems like in the last three examples can be stimulated to build the situation 'in reality' with blocks and other tangible material. In the long run, however, they are asked to work completely through their own imagination.

Problem situations here are essentially different from those in the previous sub-section *What I see*. In this case students have to talk about *how* they see it. This involves a great deal of reflection about the situation in which they find themselves, as they have to move from what they see with their own eyes to what they see with their minds' eyes. This creates a situation in which they might become aware of certain mathematical ideas and methods.

The developers of realistic mathematics education discovered that the distinction between *What I see* and *How I see* are extremely important, as they involve different kinds of identification. The first case is the most subjective. The students either describe what they see themselves, or identify with an observer. In the second case they have to reflect on the situation of the observer; identification with the observer is not enough. They have to imagine that they look on themselves from above, or from aside. One could say, that they now have to identify with two persons, one who looks and one who looks at the looker. In the first case they are part of the situation, in the second they reflect on the situation in which they are participating.

Another difference is that to get from *What I see* to *How I see it* involves a great deal of visual analytical thinking. To promote this, we have to provide the students with dynamic analytical tools, which we shall discuss in the next sub-section.

Teaching about space and shape starts with experiencing space and in particular with *What I see*, because this situation is the nearest to the personal experiences of the student.

3.3.1.3 Tools for How I see: sight-lines and sight-angles

Cat and mouse

Here is a view from above of a cat and a mouse that tries to hide (Figure 8).

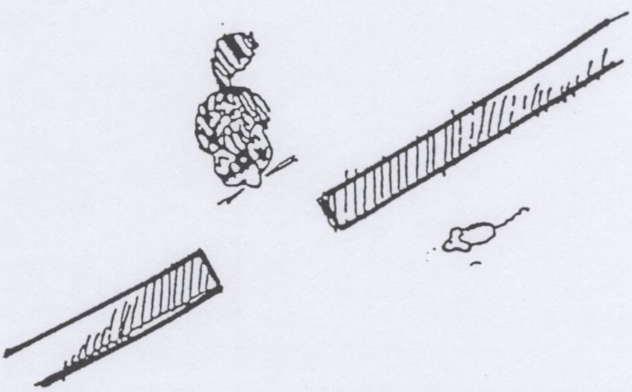


Figure 8. Cat and mouse.

- Can the cat see the mouse?
- Where can the mouse better not be?
- If the cat cannot see the mouse, from where should it be possible to see it (assuming that the mouse does not move, of course).

In the first of these three situations, students become aware that they can draw a line from the cat's eyes to the mouse. If this line intersects some objects between the cat and the mouse, then the cat cannot see the mouse. This line is an imaginary half-line starting from the eye. It is called *sight-line*. The idea of using sight-lines appeared to be a strong tool for solving problems about the relative position of objects in space. It is a first step of mathematising problems that have to do with *How I see*.

In the second situation (where would the mouse be, if the cat could see it) one can draw many sight-lines from the cat through the hole in the wall. Each of them indicates places where the mouse better not be. Together these sight-lines form an angle, which is called the *sight-angle*. (We shall not discuss here whether it would be better to imagine the sight-angle as a wedge, as is suggested here, or as two half-lines with the same end point.) Objectively the use of sight-lines would be enough as a tool, but in practice, the idea of sight-angle appeared to be useful, probably because it joins all the relevant sight-lines into one new object and as such it makes the problem situation more easy to survey. If transforming the situation into mathematics by drawing (and imagining) sight-lines, the mental construction of a sight-angle might be seen as vertical mathematisation.

The third situation (where should the cat be in order to see the mouse) looks identical with the first two at the first sight, but in fact the problem situation is completely different. At first students try to let the cat walk around until she sees the mouse. To do that, they draw several pictures of the cat on the map and in each draw the sight-line from the cat to the mouse. The problem can, however, be solved more accurately by drawing sight-lines of the mouse. This involves *a change of perspective*. Instead of identifying with the cat, as the problem suggests, one has to look at the problem from the perspective of the mouse. In the long run students learn to use this change of perspective as a useful method for solving problems. In this case the solution is to draw the sight-angle from the mouse through the hole.

Students who have difficulties to imagine the cat and mouse situation can be helped by playing the cat and the mouse. Others might find it useful to use computer programs like CABRI or INVENTOR in which it is possible to drag points.

3.3.1.4 Resume

In describing space there are two different kinds of relation between the objects that are observed and the observer. The first relation is direct, subjective and involves reflection about what the observer sees. The second is indirect, more objective and involves reflection about how the observer sees.

In the second relation, some important tools can be used to describe the situation: sight-lines and sight-angles.

In solving problems about sight-lines and -angles, one can use the method of change of perspective.

It is worthwhile to note that the situations become gradually more and more dynamic (like the moving cat and the moving mouse). Sight-lines and -angles are tools to handle the dynamic relationships between objects.

Problem situations can be materialized with real objects. Sight-lines can be materialized with pieces of cord.

3.3.2 From reality to drawings and back

In order to learn to solve problems of space and shape students gradually must get acquainted with mathematical tools. In the previous sub-section (*Experiencing space*) we described a first step. With the help of new tools, sight-line and sight-angle, students got a first experience with other mathematical tools like view-from-aside and view-from-above, which now must be developed and formalized so that they can be used both to describe situations through drawings, and to get information from drawings about situations.

We shall now shortly describe some of such tools. What kind of tool could or should be used depends very much on the given problem situation. In many cases a combination of several of these tools are necessary. In the next section (3.3.3) we shall give some examples.

- The view-from-aside as a more or less informal tool has to be formalized as *orthogonal parallel projection*. Depending on the problem situation the directions of several projections of a certain situation are not necessarily orthogonal to each other (see below in figure 9).
- The view-from-above can be formalized either as *vertical orthogonal parallel projection* or as a *map*. Both can be used to give information about depths by way of height lines (depth lines in the case of sea maps) or numbers that indicate height or depth. A map can be the same as a vertical orthogonal projection of the situation, but most often it is not. On a map objects are often not given as their projection, but as symbols, such as roads on a road map, buildings of special interests on a city map, buoys and light houses on a sea map.
- When the problem situation needs an overview, one can use several other tools. In due course students learn about *perspective, oblique parallel projection* and *orthogonal* either *orthogonal* or *oblique* (see figure 9). (N.B.: The term 'perspective' is not used in the same way in different countries. Here it used as a special case of central projection where the observer's eye – or the camera's lens, is the centre of projection and the image screen stands vertical. In other countries, like France, the term is used as a general form of projection, so that then a distinction has to be made between parallel perspective and central perspective. In that case the term 'parallel projection' is often used for oblique parallel projection.)

– A special case of orthogonal projection is *orthogonal axonometry*. (N.B.: Imagine a set of orthogonal axes in the usual way: two horizontal axes– the x-axis and the y-axis, and one vertical z-axis. Imagine a plane τ that intersects the axes. Space is orthogonally projected on that plane. This kind of projection is called orthogonal axonometry. When the plane τ is adequately selected, it gives a more natural view of objects compared to the traditional oblique projection.)

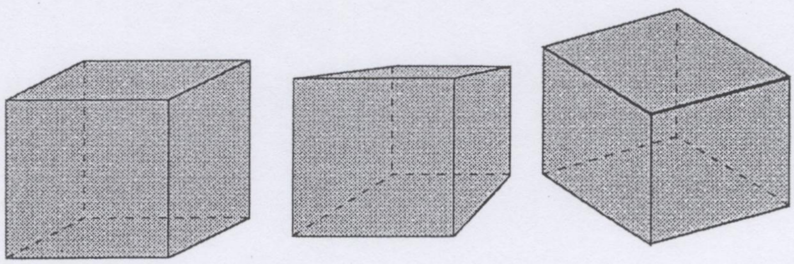


Figure 9. Three cubes in oblique parallel projection, perspective, and orthogonal axonometry.

– *Perspective* as a special case of central projection is a useful tool to get information about the general shape of an object and about the relative situation between objects. When we want to get or give more detailed information, we need other tools, like parallel projection, cross section and others. A more extensive discussion on perspective can be found in 3.4.
– Other drawing tools for describing space and shape are the *net* and the *cross-section*.

Learning to get acquainted with all these mathematical tools can and should be supported by the use of material models and instruments. Some of the instruments, such as a ruler and a pair of compasses, are well known, but others appeared to be very useful, like cords, poles, height meters, angle meters and the look-through-window.

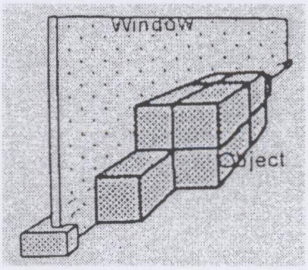


Figure 10. Look-through-window.

The look-through-window can also be used for getting acquainted with orthogonal parallel projection and perspective. In the last case a revolving door or blackboard and a piece of cord (as sight-line) may also be used effectively.

Many of these instruments can be constructed by the students themselves as a part of their explorations.

There are now good and useful computer programs that help the students to explore space and shape. It should be a serious consideration in what learning stage these programs are to be used, because in some ways they are already abstractions. When students are in the beginning of their explorations, it might be more effective to let them first work with objects (including themselves) in the real space.

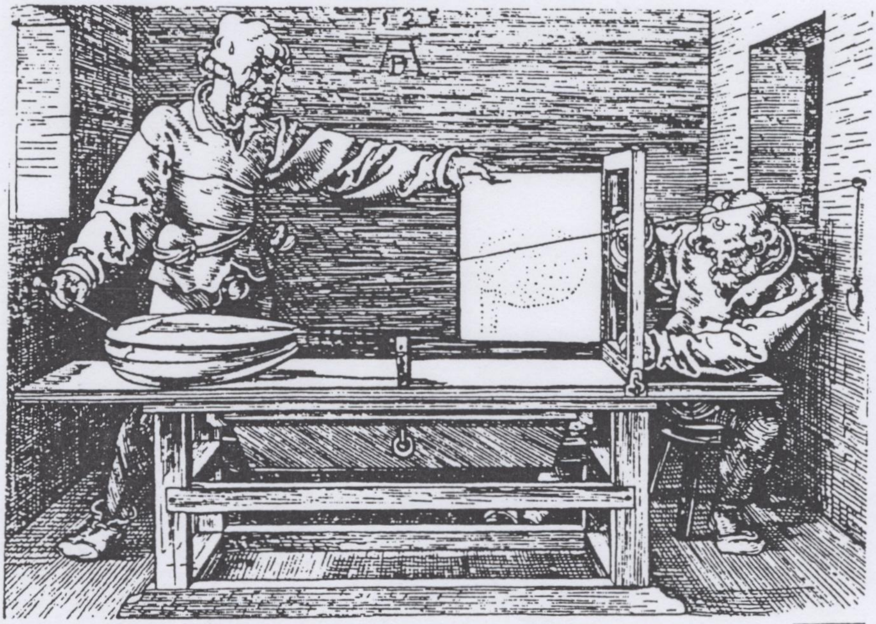


Figure 11. Albrecht Dürer, woodcut taken from the third edition of the book: The History of Art by H. W. Janson. Harry N. Abrams, Inc. New York. Prentice-Hall Inc., Englewood Cliffs, N.J.

3.3.3 Examples

A systematic and extensive description of the use of the above mentioned tools would take too much room for our chapter. Therefore we can only give some examples, chosen from the new curriculum.

Describing space with orthogonal projection: A farmhouse

Students are given a (perspective) picture of a farmhouse (Figure 12) and are asked to draw views from front, right and above like in the singer situation.

In the beginning exercises like this are concentrated on the ideas of the different views and therefore the measures of the objects and the distances between them are less important. This is the reason why these measures are not given. (Students who want to know the measures are encouraged to make their own estimates.) After grasping the ideas, they also learn to draw according to the correct measures.

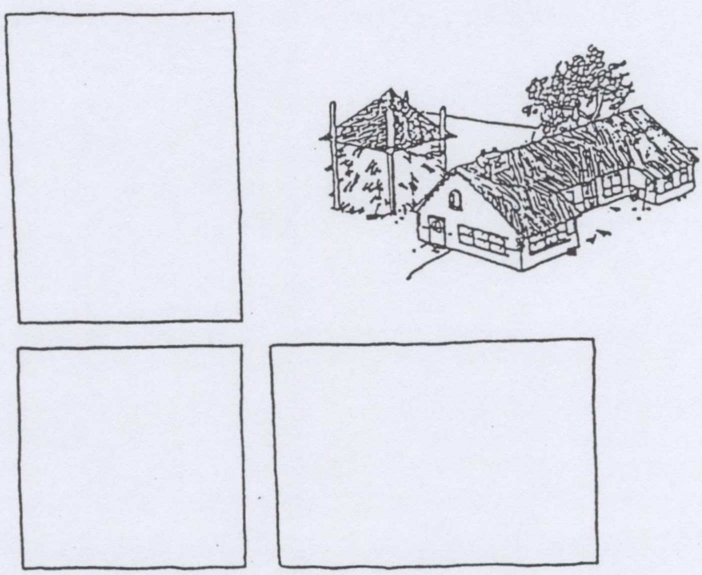


Figure 12. Farmhouse.

Getting information from orthogonal projection: Spying on an island

Side views are not necessarily 'straight' from front, left, or right. One can look at a situation from many angles. This becomes clear in the island example (Figure 13).

Through binoculars, you see from the north and from the south-east, that there are radio-masts on the island. On the map of the island, mark places of the masts with points.

Also draw what you can see through binoculars from the south-west.'

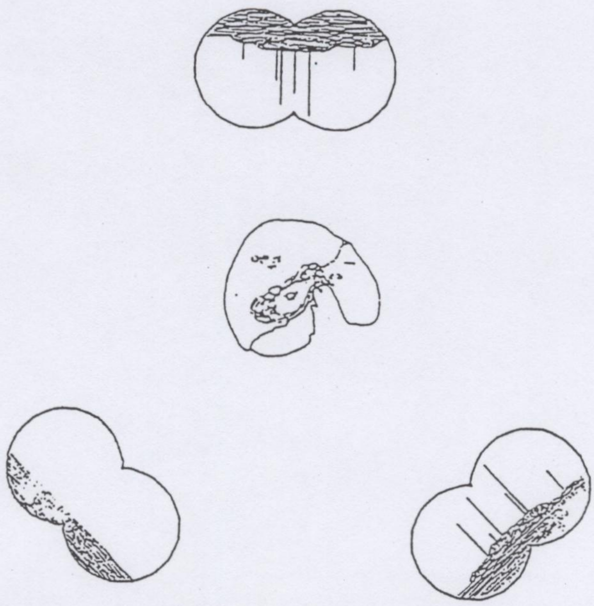


Figure 13. Spying on an island.

Using a map to describe relative positions: Ship on the river

There are beacons on the banks of a river. A ship has to navigate in such a way as to evade sand banks (Figure 14). The captain has to follow a course in which he sees two beacons as one. He has to change course as soon as he sees the next pair of beacons as one. Draw the course of the ship on the map.

A view from above in combination with numbers: Cube building

Students are told: 'Here is a view from above (Figure 15). It is a building, made from cubes. The number in each square tells you how many cubes. Draw views from two sides'.

Long before the introduction of the concept of height-lines students are prepared for it with exercises like this.

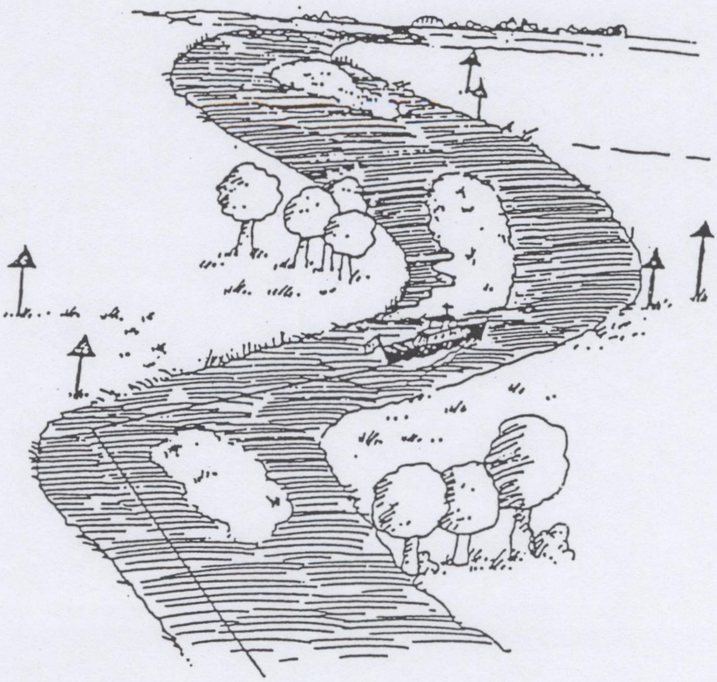


Figure 14. Ship on the river.

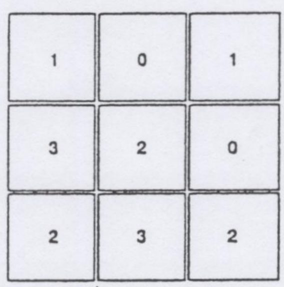


Figure 15. Cube building.

A view from above in combination with measures: Height lines

A country map as a view from above does not tell us anything about heights and depths. Therefore, if the map maker wants to give information about height and depth, this must be done by writing numbers at critical places, indicating the relative height or depth of those places (Figure 16).

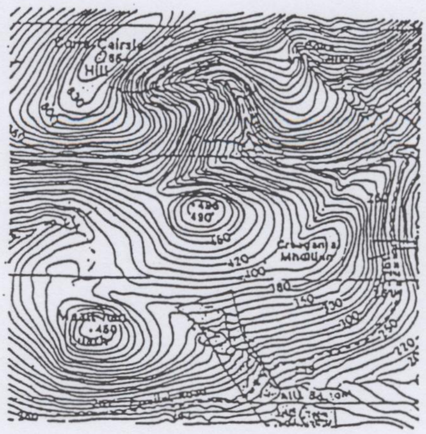


Figure 16. Height-lines.

Using height lines is also a good preparation for functions of two variables, either in the context of space and shape, or in a more algebraic context.

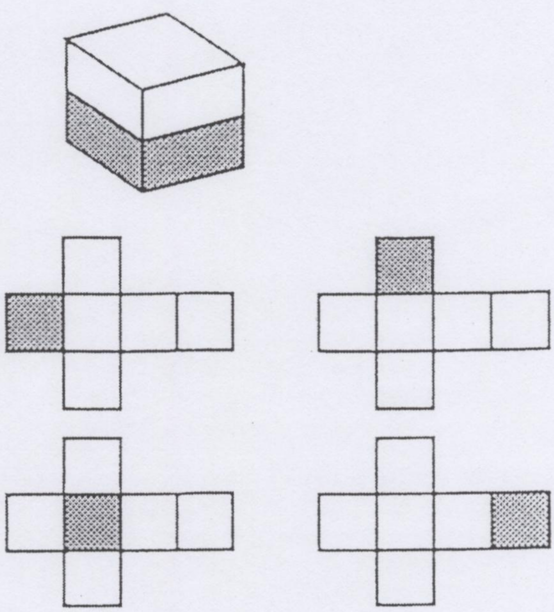


Figure 17. Black bottom.

A means to describe real measures: A net

Descriptions are not always projections. There are other possibilities (Figure 17).

The lower half of the cube has been painted black. Of each of the four nets, the bottom side is already black. Students are asked to finish them with the right blacking.

The net is not so often used as a description tool. More often as a tool to get information from it or for making tangible models.

Cross section as an 'inside' view: Cutting a pear

In this example students are asked to pair the cross-sections that divide the pear. Exercises like this can be used as a first introduction to the subject. In this case the picture is drawn directly from the object in reality. In most cases, however, this tool is used for extracting information *about* an object and *from* another picture, like in the next example.

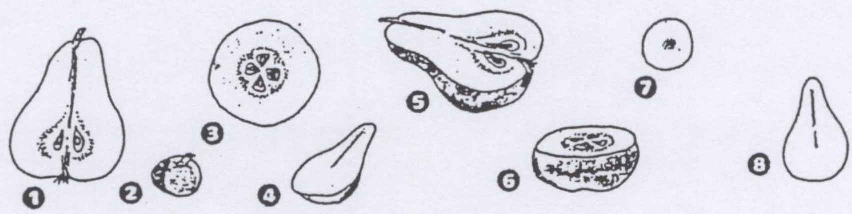


Figure 18. Pears.

Combination of a map and a cross section: The real form of a mountain

'We want to know more about these mountains. Therefore we make a (vertical) cross-section through the tops of the mountains. Draw that cross-section.

Ice on the top melts. What is the most likely route downwards, that the melting water follows?'

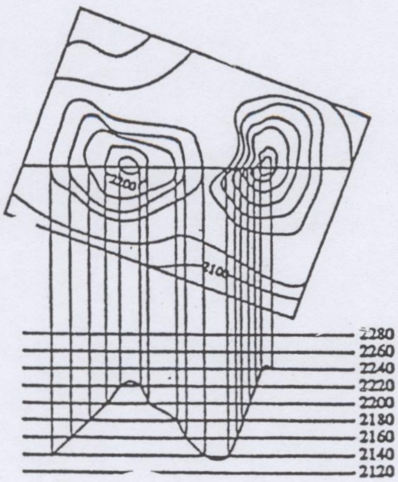


Figure 19. Mountains.

3.3.4 Other means to describe space and shape

Description by drawings is a powerful means. When talking about space and shape, it might seem to be the most appropriate, but that is not always the case. Depending on the problem situation, other means of description should be added to, or even replace drawings, such as material models, geometrical properties, measures and distances, patterns, functional relation, etc. Here are some examples.

Geometrical properties as tools for describing space and shapes

Here are two examples

- 'A strange building has three walls. Two walls make an angle of 70 degrees and the third is a half-cylinder'.
- 'A parallelogram is a quadrilateral with two pairs of parallel sides'

Patterns and numbers: Gold bars

'A thief managed to open a safe with gold bars. This is what he saw. How many bars are there in the safe?'

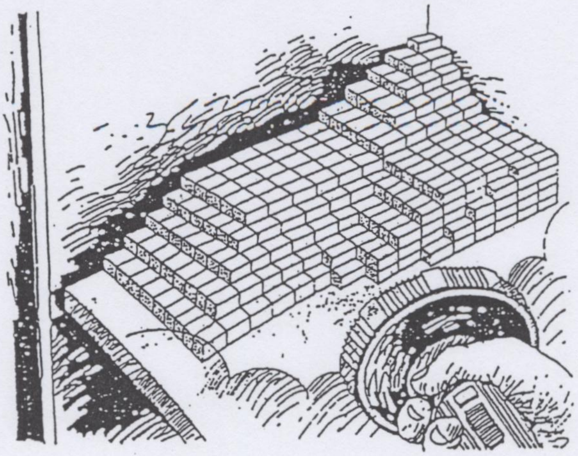


Figure 20. Gold bars.

Scale and measures: The barn

'Draw views from the front and from the left. How many square tiles with side of 0.5 meter are needed to pave the floor?'

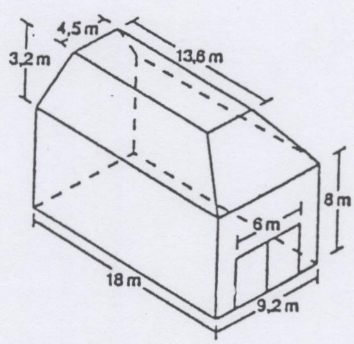


Figure 21. A Barn.

Using algebraic functions: A cooling tower

'A cooling tower has a height of 22 meters. In a orthogonal *xyz*-system its surface can be described by the equation $x^2 + y^2 = (z-11)^2 + 20$. What is the form of cross sections of a plane that is parallel to the horizontal xy-plane? Which of them is the smallest? Draw a cross section through a plane that contains the vertical z-axis.'

3.3.5 Remarks

(See also the discussion about micro-space and meso-space in 3.4.)

What about proofs?

Geometrical properties are the core of the traditional geometry curriculum. One might wonder what is left of this when much attention is given to direct relations between space and shape in reality and their mathematical descriptions. What is left of formal proofs? Geometrical properties are however still an important part of the new curriculum, both as a tool to find information about space and shapes, and because they are the very elements in formal and common sense reasoning. When students are actively working together with each other there often comes a need for common sense reasoning. How else can one convince one's partner without using arguments of force and bullying? All, or almost all, the examples that were given before require common sense reasoning. In a very few of them one can take refuge to well-trained but poorly understood skills. As such, this kind of activity is also a good base for learning formal reasoning in the long run. Gradually students learn to trust not only what they see, what they measure, calculate, draw, but also on their logic reasoning. They also learn when they can not trust what they see. For example: In a drawing one can see intersecting lines that in reality do not intersect at all. In case of doubt, one can draw views from other positions, one can use another kind of projection, one can draw cross sections. But all these methods, if they give a satisfactory result at all, are often much more complex than a reasoning in which well known properties are used.

Reality, drawings and mental image

When students begin to learn to relate space and shape with drawings there will not be a direct relation between the real object and the drawing. Any relation like that is embedded in a mental image.

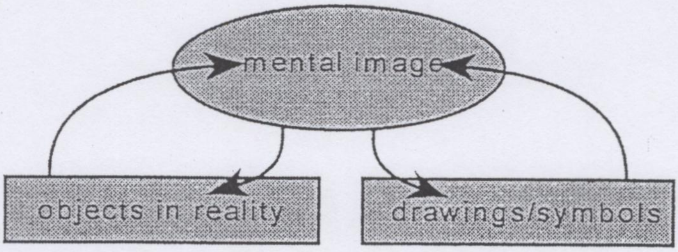


Figure 22. Mental image.

Without a mental image one might be able to make a correct drawing, but only when one knows rules about what to do in a certain situation (or when one is told exactly what to do). Expert draftsmen need such rules, because they cannot make a drawing of a complicated situation if they have to imagine every little detail and ask themselves how to draw that. Beginners, however, do not know yet the signs and symbols that are used in depicting certain concepts. They do not yet know the conventions that are in drawing objects. All they are able to do is to make a direct relation between reality and their own mind, like young children who learn to recognize objects from pictures. Only in the long run are they able to make pictures of reality and to interpret pictures as an image of reality.

3.4 Space geometry and their representations by diagrams

When space geometry is concerned, some questions arise soon: What are the place and role(s) of diagrams in their teaching and learning? How do students 'read' usual diagrams? It is a fact that mathematics teachers often consider them self-evident, but is this evident? ... Since diagrams are the usual – and most often the only, mediation for spacial situations in geometry, the question cannot be neglected and this is the reason why a subgroup of the DIDIREM team (Paris-7 University) undertook a research on that subject. This research examplifies the main question linked to space geometry diagrams and proposes an explicit taking into account of their learning in secondary school.

After a decade or so during which diagrams were practically banished from the teaching of geometry, the situation has changed much. From some years on, and certainly as a consequence of several researches in mathematics education, the official programs emphasize the representations of space and manipulations on and with these representations, in elementary school as well as in junior and senior high school.

For instance, the programs say that, at elementary school, children should manipulate and represent solid objects, but they do not say which kinds of representation have to be involved. Manipulation seems indeed to be a good help for the learning of geometry: a comparative study between French and Japanese junior high school students (Grenier & Denys 1986) about axial symmetry showed a real difference in performance in favour of the Japanese. The authors attributed this difference mostly to cultural factors, among which the practice of origami (traditional paper folding) would play a significant role.

At high school level, a type of representations which should be used to represent three-dimensional objects is mentioned: it is parallel perspective. (See a remark in 3.3.2 about the use of the words 'perspective' and projection.) Two French researches (Parzysz 1988), (Colmez & Parzysz 1993) have shown this type of representation has been shown to be in adequacy with the conception of the students of this level about diagrams: they consider most important the preservation of some properties of a given configuration in drawings representing it and especially parallelism. In fact, an explanatory model of the relation of an individual with diagrams representing spatial objects has been proposed; it is based on the antagonism between representing the object as it can usually be seen and representing it in its essential characteristics (i.e. in our case, geometrical properties); so, representing a given geometrical object by a single diagram requires more often to find a compromise solution between these two options. (N.B.: the same principle applies to the way one 'reads' such a diagram as well).

Thus parallel perspective seems a good tool for representing space in school geometry. We may notice that this includes nearly all the diagrams usually found in geometry textbooks (orthogonal views, perspective sketches) even if their belonging to this type of representations is not mentioned.

Unhappily, the order in which the objects are studied in junior high school shows clearly that the problem of representing them is not taken into consideration (for instance, nothing is said about how to draw a circle in perspective – i.e. an ellipse, which seems useful, if not necessary, when studying the cylinder and the cone, as well as sections of a sphere by planes). This unclear situation causes a great confusion in textbooks: in fact, most of them consider parallel perspective a mere set of conventions (which it is not, since it is a mathematical transformation; to be precise: a parallel projection on a plane), and they give some rules for drawing (in fact, most of them are geometrical properties of the projection: such is the case for the preservation of parallelism and ratio of distances on a line).

To end with, at senior high school, a combined use of three-dimensional models and plane representations is required, and students must be trained to a frequent use of sketches in perspective, using dotted lines (grade 11).

All this shows that, in the French geometry curricula, the learning of space and shape includes – at least in the official texts, a frequent use of manipulations and of plane representations. This seems to be a sensible evolution, but, in spite of a recurrent mention of the representation of these various objects, the learning of such representations is never clearly taken into account in the programs.

However, giving a real geometrical status to diagrams used in space geometry could be a means to make them an efficient tool for solving problems. Besides, this problem of the representation of space is indeed a real one, since nowadays students are more and more confronted with varied images, in their daily life as well as in school. Moreover, it could constitute an interesting (if not easy) subject of geometrical study for high school students.

3.4.2 A French research study on the teaching of both space geometry and plane representations at the high school level

3.4.2.1 The basic ideas

Three directions can be distinguished in the learning of space geometry at the high school level: *space* (i.e. space in which we live and move), *geometry* (as a model of this space, i.e. a set of definitions and rules used to solve spatial problems) and *representation,* which can be three-dimensional (models) or two-dimensional (sketches, drawings, diagrams...). These three directions interact together in learning, but in France – as seen above, the last one is mostly ignored, or at least not given a real place and status.

This is the reason why the team took the learning of plane representations, together with the learning of space geometry, as a subject of study. The experimentation began in 1987, in several classes of 11th grade. This research refers explicitly to two of the three perspectives indicated at the beginning of this paper:

- three-dimensional models were used in order to simulate a physical phenomenon (shadows) and, together with two-dimensional diagrams, to help its understanding (perspectives 1 and 3, where 1 is a prerequisite for 3).
- These models were also used to introduce the elementary geometrical concepts in space (lines, planes, their possible relative positions...), and then to construct a theory of space geometry (perspective 2).

The main ideas of the study were the following:

1) In opposition with plane geometry, plane representations are not isomorphic with the spatial configurations which they represent, whereas three-dimensional models are; thus, to make learning easier, such a learning could profitably rest on a frequent use of models, even in the case of senior high school.
2) Nevertheless a student's relation with a spatial situation involving small objects (*micro-space*) is different from his or her relation with a situation involving bigger objects (*meso-space*). More precisely, in micro-space the situation is completely controlled by the senses (sight, touch), any point of view is easily available and actions on the objects do not require much effort; but in meso-space control and actions are not so easy and one has to replace, to some extend, action by reflection (Brousseau 1986). Thus micro-space should make the formulation of conjectures easier, whereas meso-space should favour resorting to argumentation.
3) A consequence of this lack of resemblance between configurations and representations is that a learning of good relations between them must have its place, including both reading diagrams and drawing them. Anyway, the need of such a learning had been inferred from a study of some mistakes and misconceptions obviously linked to representations. For instance, the usual representation of a plane by a parallelogram induces students – even at upper grades, to believe that two planes can intersect in a single point or that they can have no common point without being parallel to each other (Parzysz 1991).
4) For the reasons exposed above (preservation of some geometrical properties), as well as for its versatility and its general use, parallel perspective was the type of representation chosen to be integrated to the teaching of space geometry.

This research planned simultaneous introduction to space geometry and to plane representations; thus, parallel projection on a plane appeared as a central geometrical concept. A process had then to be imagined, put into action and assessed, which should encompass the learning of the usual geometrical concepts and properties, as well as rules of drawing. The typical scheme on which such a process was built is the following: a situation – implying one or several conjectures, is discussed by the whole class, under the direction of the teacher (scientific debate), which leads to a (more) precise formulation of the question(s) to be solved. When the problem has been solved, the main results are taken into account in an official way (i.e. writing them down in copybooks as definitions, theorems...) and finally the students are then asked tasks requiring use of these results as tools, together with 'old' tools.

3.4.2.2 The problem situation

A previous research by the same team had taken the study of shadows cast by the sun on the ground as a basic situation (Colmez 1984). This choice was justified by the fact that this is a very common phenomenon, about which students are convinced to know everything; the aim of such a simple situation was to cause an imbalance in their minds, when some results they all thought true would prove to be wrong (following Bachelard's idea:

> 'one knows against a previous knowledge, by destroying an ill-made knowledge'
>
> (Bachelard 1983)).

However, in that first research, some students were reluctant, from the start, to accept that the rays of the sun falling on an object were parallel ('they diverge from the sun'). Thus, the new research started with the study of shadows cast by an electric light bulb, with the purpose of making the spontaneous conceptions of the students evolve towards the concept of parallel projection when the bulb was replaced by the sun; then a new situation could be based upon shadows cast by the sun, and studied in similar conditions.

The use of three-dimensional models had to be emphasized (see above); then, the evoked situation (meso-space) and the materials (micro-space) were designed in a double purpose:

- help the students imagine the situations and formulate conjectures (micro-space);
- enable them to experiment and justify (meso-space).

In this process, a crucial point was the move from shadow to perspective; to solve this problem, the 'isomorphism' between the materials and the device – in use from the Renaissance on, known as Dürer's window was used.

The evoked situation (shadow, cast by a street lamp, of the upper side of a cube lying on the ground) belongs to meso-space; this situation is materialized by a model (micro-space).

This situation is intended to make the imbalance appear clearly, since the answers given are very diverse, whereas the correct answer (the shadow is a square *in every case*) is given by less than one student out of ten. But nothing is said by the teacher about the correct answer, which leads among the students, to a socio-cognitive conflict. To solve this conflict, a simulation is realized with the help of the model. The observation of the device leads to a more precise conjecture (Is the shadow square in any position? One cannot be sure, because the imprecision of the situation).

The students have now to answer this new question, and, because the above pragmatic justification cannot be of any help, they have to make use of reasoning. The deductive proof(s) comes(come) out of a collective discussion. To support their reasoning, the students start of course from what stands in front of their eyes, i.e. the materials; to explain their ideas, they have to make use – in a quite intuitive way, since they are just beginning to study this domain, of space geometry rules (like for instance: 'when a plane intersects two parallel planes, the intersections are parallel lines').

They are also led to draw some diagrams on the blackboard, which are in fact plane geometry drawings, since they represent what is in a particular plane. Thus their deductive proofs can integrate, as official tools, plane geometry results already known. We can notice that the use of drawings *at a given scale* (1/10), based on measures of the materials and coordinated with one another was emphasized with younger students (10th graders), because this appeared to be an important factor of involvement in the task.

As we said, the justifications given by the students contain necessarily a part of implicitness about space geometry, since they are just beginning studying it. Starting from their statements and actions, the teacher asks them to explain their ideas: some facts seem implicitly 'evident' to them, but they have to be made explicit to become geometrical rules (definitions or theorems). For instance, starting from a drawing made by a student, a discussion started up how to determine a plane. Let us notice that, up to this moment, several domains have been interfering in the processes:

- real space (i.e. the materials)
- geometrical space (points, lines, planes...)
- plane geometry (theorems)
- (partial) representation of spatial situations (sketches, drawings at a given scale...)

3.4.2.3 From shadow(s) to perspective

The lamp shadow was intended to be a materialization of a theoretical means for representing space (precisely: central perspective). However, though the aim was *not* the teaching of this kind of representation, establishing – in a general way, a strong link between shadow and perspective appeared as a necessary element of the processes.

A first step was made when the students were asked to draw the *complete* shadow of the cube. This proves to be an easy task for them and at this moment, several students remark that the shadow 'looks like the drawing of a cube in perspective'. The resemblance between shadow and drawing is then emphasized in another way, thanks to the 'isomorphism' existing between the materials presented at the beginning of the study and the device known as Dürer's window (see sub-section 3.3.2).

Thanks to the model belonging to micro-space, the isomorphism can be made evident by turning the materials on one side – the board now being vertical instead of horizontal, in order to show the correspondence with Dürer's window.

The move towards sun shadow is based on a collective investigation about the properties of the shadows cast (virtually) by the light bulb; more precisely, the question of knowing if the shadow (on the ground) of the middle of a vertical stick can be the middle of the shadow leads to the condition of the light bulb being 'very far', or 'at infinity' and thus to the bulb being finally replaced by the sun.

The question is now to study the shadow of the cube cast on the ground by the sun. Hence another device has to be used as a micro-space model of the situation, to simulate the phenomenon. The progress of the next sequence is very similar (except of course for the details) to the previous one; then, the link between shadow and (parallel) perspective is realized once more by turning the materials on one side, since the plane of the picture is traditionally supposed to be vertical (See also Caron-Pargue 1979).

Then a necessary phase takes place: institutionalization, i.e. noting down the most important things which had been encountered, as well in space geometry (definitions, theorems) as in the representation of solids (geometrical definition and properties of the parallel projection on a plane, graphic conventions: dotted lines, representation of a plane by a parallelogram...).

This phase of mathematisation is followed by varied activities, among which some exercises, which seem at first to consist only of the completing of a diagram, are in fact genuine *exercises of space geometry*. For instance the following (Figure 23):

> The horizontal plane (H) and the vertical plane (V) intersect along (D); point a is the orthogonal projection of point A on (H), B and C belong to (V). Draw the intersection of plane (ABC) with (H).
>
> Succeeding in such a task requires the use of *spatial geometrical knowledge* (any point of (V) is projected in (D); the invariant points of the projection are those belonging to (H)); this includes properties of parallel perspective (parallel lines are drawn parallel).

In fact, the realization of a correct diagram requires a full understanding of the spatial situation, but it is also necessary to make use of graphical conventions (dotted lines).

The way in which the students solved these tasks showed that giving *a full mathematical status* (mathematisation) to the representation of space helps them to give some sense to the tasks: making a correct diagram is not just following arbitrary rules, but solving a geometrical problem *and* following rules (partly geometrical and partly conventional). Most students were able to use parallel perspective as a useful tool in spatial problem solving, which enabled them to make conjectures, as well as to prove them. Moreover, the use of partial diagrams (sections, views) led them to reinvent plane geometrical knowledge.

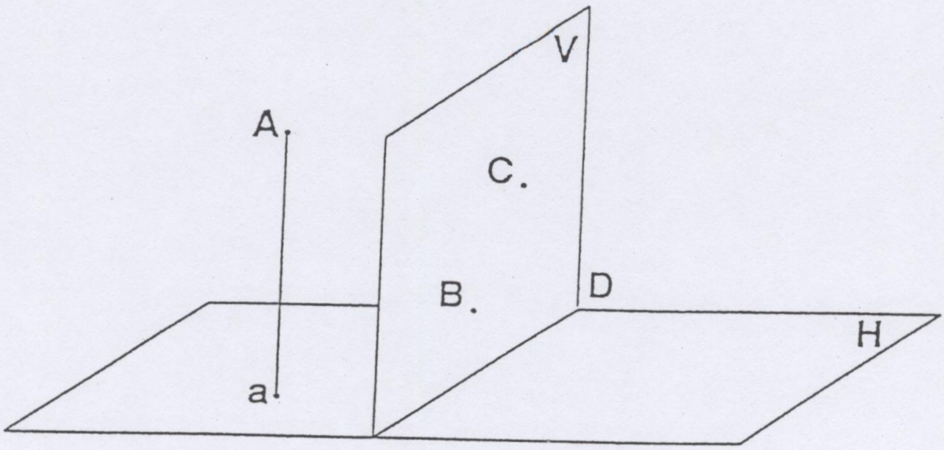


Figure 23. Understanding it geometrically.

To end with, the situations chosen to introduce the representation of space in parallel perspective, although found rather difficult by the students, proved to be of great interest for them (as the liveliness of the debates and a final questionnaire showed), and they were most surprised to see that such simple situations could nevertheless be puzzling (a major reason for this being that their knowledge about shadows regards mostly shadows of vertical objects, whereas in this case they were asked to imagine the shape of the shadow of a horizontal square).

The assessment of this research confirmed mostly the analysis that had been done *a priory*:

1) Parallel perspective, in its varied forms (cavalière perspective, views, etc.) proves to be well suited to high school students, because
   - on the one hand, important properties of configurations (parallelism, ratio of distances on a line,...) are preserved on the diagrams,
   - on the other hand, it retains an acceptable resemblance to the objects represented, and helps the constitution of good (i.e. precise and well-structured) mental images.
2) Making the rules of drawing explicit helps the students to realize useful diagrams, since this gives sense to their graphical actions.
3) Teaching the students to use varied types of diagrams and to move easily from one type to another helps them in the solving of spatial problems, since they can use the most suitable drawings to solve a given task.

4) The constant use of three-dimensional models showed its usefulness: at beginning of the experiment they were a real support for reasoning, enabling the students to choose a good point of view; but, as time went by, most students became able to imagine clearly the spatial situation, without having the materials before their eyes; they seemed thus to have developed efficient mental images of these situations (which is indeed an important aim to be reached in the teaching of space).

Of course, this experiment must be considered an example of what can be done in the direction of giving representations of space in geometry a real status, and of course other situations could be used for the same purpose. (N.B.: The same French team is now working on the teaching of space geometry at junior high school level, and more especially on 'round bodies' (sphere, cylinder, cone) and on circle in perspective (ellipse).)

CONCLUSION

At the beginning of this paper we described three perspectives according to which visual education can be examined. We choose to focus on the first perspective, *Interacting with real shapes in space*, because we consider it as a very basic one, while on the other hand it is the more neglected one of the three.

To clarify our purpose we gave three examples of projects. They are different: in their immediate goals, in the approach according to which the sequences of activities were developed, in the age of target populations, etc.

However, the common features that they do have are the motive for including them in this chapter. We see their most important contribution to visual education in the following:

a) In the three examples *shapes and space*, or better: *shapes in space,* are the starting points of the learning-teaching activity.
b) A main goal for students is to feel the need for, and to construct mathematical actions and tools, with which they can better examine and analyse the relations between and within visual objects. In other words: students are led towards mathematisation of the visual environment with which they interact.
c) The mathematical tools and actions are very rich. They include basic actions like identification or analysing the components and properties of visual entities. But they go beyond that and consider dynamic relations and high level orders between the visual entities in space.

These mathematical tools and actions can be seen, for example:

– In the *Agam project*,: The dynamic relations between shapes which come out from the mathematical actions on shapes while children examine and create patterns (periodical sequences of shapes). Or, the analysis of the order between shapes via the direction criteria (horizontal, vertical and oblique).

- In the second example, *Shape and Space and Reality*: The mutual dynamic relations between some seen objects, *What I see,* and between objects and the observer, *How I see*, and the process of mathematisation of these relations.
- In the third example, *Space geometry and their representations by diagrams:* The dynamic relations between objects in space and shapes attached to them, like an object and the changing shape of its shadow.

We did not mentioned computerized tools, although they can have a major role in the mathematisation process of the visual environment. (These tools are discussed by Balacheff and Kaput in this volume). Here we want to mention two features that seems to us most relevant to shapes in space.

The ability to create a sequences of shapes out of one shape, by the drag mode in software, enable the students to concentrate on the invariant properties of such sequences of shapes. This can be done – for example, in the Cabri Geometre (Laborde & Sträßer 1990), the Geometer Sketchpad (Key Curriculum Press, USA), or the Geometry Inventor (1994). Laborde (1993) claimed that: 'Geometrical relations can be visualized as invariants under the continuous movement of the figure', and that: 'A further dimension is added to the graphical space as a medium of geometry: the movement' (p. 56).

In order to produce geometrical shapes on the screen that will not collapse under the graphical actions like the drag mode, the student has to describe it geometrically in an explicit way (Laborde 1993, p. 54). This again stress the geometrical properties of the shapes which are tested by the immediate feedback that students get from the computerized environment.

We see our task as mathematics educators to provide students both with a meaningful examples of 'space and shapes' and relations, and to help students to use mathematical tools needed for interacting and controlling this space construct in some way.

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