ZVIA MARKOVITS[1] and RINA HERSHKOWITZ[2]

RELATIVE AND ABSOLUTE THINKING IN VISUAL ESTIMATION PROCESSES

ABSTRACT. This study has two main goals: (1) to investigate the processes involved in visual estimation (part I of the study), and (2) to investigate the processes of judgment in visual estimation situations, which mostly involved proportional reasoning (part II). The study was conducted with 9-year old children in the third grade. Four strategies were expressed by the children in visual estimation situations. Exposure to a unit in the Agam project, designed to enhance visual estimation capabilities resulted in changes in the children's strategies. These changes reflected the processes by which children overcame their limited ability to process visual information. The development of proportional reasoning was investigated through a series of judgment situations. Although, as was expected, most of the children showed an additive behavior, these situations stimulated some children towards qualitative proportional reasoning, where easy/difficult considerations played an important role.

1. INTRODUCTION

The main goals of this study were:

1. to investigate visual estimation processes (Part I), and
2. to investigate processes of judgment in visual estimation situations, which mostly involved proportional situations, (Part II).

The two parts of the research were done with 9 year old children.

1.1. *The study problem situations*

Two types of problem situations form the basis for the two parts of this study:
For part I: *Estimation situations*. Flash cards bearing various numbers of dots (see Figure 1) were presented for a short period of time to a child. The child was asked to evaluate the number of dots and then to explain how s/he arrived at that number.
For part II: *Judgment situations*. The subject was asked to judge the quality of several estimates, and to explain how she or he arrived at that answer (see Figure 2). These situations were mostly proportional reasoning situations.

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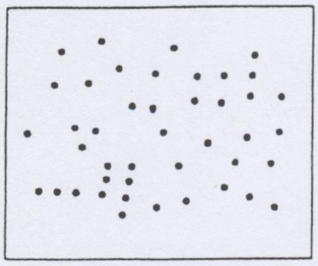
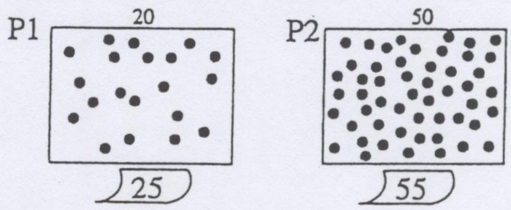


Figure 1. A dots flash card.

Two dot pictures were shown to Noa for a short period of time. There are 20 dots in picture 1 (P1) and 50 dots in picture 2 (P2).



Noa said that there are 25 dots in P1 and 55 dots in P2. Was one of Noa's answers better than the other one, or were both answers equally good?

Figure 2. A judgment situation.

The above estimation and judgment problem situations involve the integration of: (i) estimation processes, (ii) visual information processing and (iii) proportional reasoning.

In the following section, we will discuss the main issues relevant to these three components.

1.2. *Estimation, visual information processing, and proportional reasoning*

There are many daily situations in which we use estimation processes. The process of estimation depends on the situation itself as well as the estimator. The final step in the estimation process is the result – the estimate, which should be reasonably close to the exact quantity. Defining whether the estimate is reasonable means that we need to be able to judge the estimate in relation to the quantity estimated. Proportional thinking is required for this purpose.

The following questions can form a basis for research which involves any type of estimation processes:

a. How good is the subject in his/her estimate, i.e., how close is the estimate to the exact quantity?
b. What was the strategy that led the subject to her/his estimate?
c. In what way is the subject aware of the quality of the estimate? How does s/he judge that one estimate is better than the other?

In each of the above three questions the effect of instruction (experience) can be investigated.

Several research studies have investigated the first two questions; the ability to estimate and the strategy involved in arriving at the particular estimate. These studies suggest that the estimate obtained and the strategy used depend mainly on the situation, the age and the experience of the estimator (e.g., Reys et al, 1982; Hildreth, 1983; Glinner, 1991).

Judging whether one estimate is better than another is related to proportional reasoning. Many research studies about issues related to proportional reasoning in general have been carried out (e.g. Karplus, 1983; Hart, 1988; Behr et al., 1992; Lamon, 1993), but few studies have investigated the relationship between proportional reasoning and estimation processes.

Our judgment situations fit the criteria of comparison problems in the context of proportional reasoning, where all four values (A,B,C,D) are given, and the goal is to judge which is true:

$A/B < C/D$ or $A/B = C/D$ or $A/B > C/D$ (Lesh et al., 1988). In our judgment situations, as described in Figure 2, B and D are the quantities to be estimated and A and C are the estimates, where $|B-A|$ and $|D-C|$ are the absolute errors. There are two sets of ratios A/B and C/D or $|B-A|/B$ and $|D-C|/D$ which should be compared in the judgment process. These judgment situations involve several types of comparisons:

1. When $B=D$, the quantities to be estimated are equal. In this case in order to judge which ratio is bigger (equal, smaller), it is only necessary to *absolutely* compare the estimates, A and C, or the absolute errors, $|B-A|$ and $|D-C|$.
2. When the quantities to be estimated are not equal ($B\neq D$) but either the estimates are equal ($A=C$) or the absolute errors are equal, ($|B-A| = |D-C|$), then it is safer to use a proportional judgment.
3. When $B\neq D$ and in addition, either the estimates are unequal or the absolute errors are unequal ($A\neq C$ and/or $|B-A|\neq|D-C|$), then proportional calculations of the relative errors must be performed.

Research studies on proportional reasoning suggest some developmental stages in the strategies used by subjects presented with proportion tasks. A pre-proportional strategy is the additive strategy (Karplus et al., 1983; Hart, 1988; Lesh et al., 1988; Nunes et al., 1993 and Lamon, 1993), where children treat the data in an absolute way. They add or subtract instead of multiplying or dividing. While the additive strategy will work in the first type of the comparison tasks, it is not appropriate for the other two. Thus as Lamon (1993: p. 58) suggested, "not only do students need to recognize the relative comparison as an alternative to their additive view of the world but also to develop criteria by which to judge which of the perspectives is appropriate in a given situation."

Hart (1988: p. 217) drew attention to studies that suggested a lack in consistency of approach by individual children. Thus, although the child used proportional thinking in one task, s/he did not use the same strategy in another similar task. Hart also pointed out that the relations between numbers used in these studies, were small integers. In addition, Hart showed that "there is evidence that young children and the less successful secondary school pupils have a sense of what 'looks right' or of what seems to be a distortion."

In a study involving proportional reasoning in computational estimation, (Sowder and Markovits, 1990), it was found that seventh graders had difficulties understanding that, in certain situation, the use of relative error would be more appropriate than the use of absolute error. In another study, Markovits (1987), found that when dealing with estimation in measurement, both sixth and seventh graders as well as, preservice and inservice teachers used absolute errors when relative errors were more appropriate. In the same study, it was found that the students were much more successful, after a brief instructional experience. Such instruction also had an impact on seventh graders, when judgment situations were in the context of computational estimation (Markovits and Sowder, 1994).

In our study, we were concerned with *visual estimation of discrete quantities*. Visual processes are part of the processes by which human beings deal with the representation of visual information.

According to Piaget's theory, "the representation of space is not a perceptual 'reading off' the spatial environment, but is the build-up from prior active manipulation of that environment" (Clements and Battista, 1992: p. 422). Bryant (1974), who continued Piaget's approach, found that humans are born with the ability of gathering information by *relative codes* only, and not by *absolute codes,* meaning that children can say that one object is bigger (heavier, sweeter, etc.) than another object, but they cannot say how big it is. With experience, children learn to use *external frames of reference* and gradually, learn to deal with absolute information. In this sense, a child can use any previous chunk of information as a frame of reference. Thus, Bryant claims that the perceptual judgment of the child is based on deductive inferences; unlike Piaget, he thinks that the child is able to infer deductively from a very young age.

In psychological research studies, it was found that when time was restricted, eight is the largest number of objects one can recall with exactness (Folk et al., 1988). In our study the quantity of objects used is much larger than 8. Thus, human perception is not enough to evaluate the exact number of objects, and some estimation processes must take place.

2. The Study

2.1. *Background*

The study was performed as a part of the Agam Project for visual education. This program was envisioned by the artist Yaacov Agam and has become an educational reality through the work of a team of researchers and educators of the Science Teaching Department in the Weizmann Institute. The project is an example of an effort to interweave the development of visual language with a process of developing visual thinking processes. Some of the program's thirty-six curriculum units introduce students to such basic visual concepts as the major geometric figures, directions, colors, and size relationships. These units make up a "visual alphabet" that forms the basis for more advanced concepts, such as symmetry, ratio and proportion and other concepts that serve as building blocks in scientific and mathematical thinking (Razel and Eylon, 1990; Eylon and Rosenfeld, 1990; Hershkowitz and Markovits, 1992).

One of the goals of the Agam Program was to develop visual estimation as an ability and as a part of visual thinking processes. Unit No. 28 in the program deals with visual estimation. The implementation of this unit was the trigger for the study discussed in this paper.

2.2. *Research questions, research population, and research design*

The goal of this study was to investigate the three main research questions mentioned above, in the context of visual estimation. In part I of the study, we investigated the two first questions. (See also Markovits and Hershkowitz, 1993):

a. Subjects' visual estimation abilities.
b. Types of strategies used in visual estimation situations.

In part II we investigated the third question:

c. Self judgment processes, or in other words, the subjects' awareness of the estimates' accuracy.

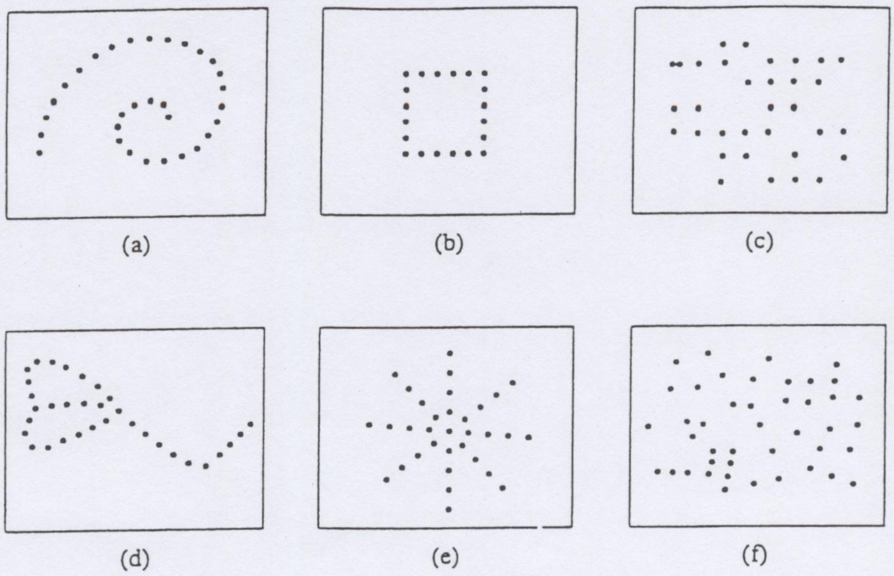


Figure 3. Flash cards for both interviews (a, b, c for the first interview and d, e, and f, for the second).

Twelve third-graders (about 9 years old) participated in this study. The children were interviewed both before and after an instructional unit designed to develop visual estimation abilities. The instructional unit did not include activities to develop estimation strategies or self- judgment processes. The interviews focused on the above three research questions, that is, children's visual estimation abilities, estimation strategies, and self judgment processes.

2.3. *The visual estimation unit (VEU)*

The main objective of the visual estimation unit was to develop children's visual-numerical intuition by providing experience in visual estimation tasks. The unit included the following activities:

i. A set of 30 flash cards, consisting of dot pictures in which the number of dots varied from 10 to 40 and the arrangement of the dots was also varied (see for example Figure 3 a,b,c), was presented to the children for a very short period of time. Each child was asked to write down how many dots s/he had seen. As immediate feedback, the teacher showed them the exact number of dots, and the children wrote down the correct number next to their own answers.

ii. Concrete objects, such as pencils, plastic circles and squares, which were spread randomly on a table, as well as pictures in children books and objects from the classroom environment, were presented to the children; The children were asked to write down the number of objects they observed. As above, the teacher showed them the exact number of objects which the children recorded next to their own answer. iii. Visual estimation activities were also conducted outside the classroom, using the number of windows in multi-story buildings, the number of trees along the road, etc.

The unit's goal was to enrich the visual experience of each child. Neither estimation strategies (research question b) nor the awareness of the quality of the estimates (research question c), were discussed during the learning of the unit. Our study investigates the kind of strategies and ideas that children develop by themselves through their own experience.

2.4. *The interviews*

In the first part of each interview, visual estimation tasks were presented. Figure 3 a,b,c, shows the three pictures presented, with 31, 20 and 32 dots, respectively. A marble necklace with 189 marbles was also presented. In the second interview, different dot pictures were presented with 33, 33, and 42 dots (Figure 3 d,e,f). The arrangement of the dots in both interviews was of the same type. The same necklace was also presented. In both interviews, the pictures were shown for a short period of time only; This period was insufficient to allow them to count all the dots. The necklace was presented for a longer period of time and the children were allowed to take it in their hands. In each situation, the children were first asked to evaluate the number of objects (research question a), and then to explain how they evaluated the number (research question b). Unlike the activities in the instructional unit, no feedback about the exact number of objects was provided to the children.

The second part of each interview included judgment situations, developed for the purpose of investigating research question c above. (A detailed description will be given in the next section.) The results will be reported according to the two parts of the research and the three research questions discussed above.

3. Results – Part I: Visual Estimation Abilities and Strategies

3.1. *Visual estimation abilities* (Research question a)

It seems that children usually do not estimate. They prefer exact answers. It was difficult for many of the children to accept answers close to the exact number as good answers. For example, a child would say "I goofed" when her/his answer was only one or two away from the exact number. Only during the later stages of learning, following discussions with the teacher that it was all right not to get the exact number, sentences such as "I *was close*" were heard in the classroom.

In general, children were closer to the exact number in the second interview than in the first. The average of the children's relative error for the three pictures was 27% for the first interview and only 14% in the second. When asked to estimate the number of marbles in the necklace, the children performed better in the second interview. These results suggest that exposure to the visual estimation unit improved the children's visual abilities.

Children were closer to the exact number when the dots were arranged in a geometrical pattern (e.g., Figure 3b and 3e).

Most of the answers were underestimates in both the pre- and post-interviews. For the dot flash cards, 2/3 of the answers were underestimates, and only 1/3 overestimates; exact answers were given in very few cases. The phenomenon of underestimation was much more common for the necklace. Here *all* answers were underestimates and were far below the correct 189. The average number of children's estimates was 65 in the first interview and 81 in the second. We believe that the main reason for this underestimation is the limited exposure to numbers of this magnitude that children at this age might have.

3.2. *Strategies of estimation* (Research question b)

We identified four different strategies used by the children.

I. *Counting strategy.* Children using this strategy counted as many objects as they could in the short period of time available and then added some more. For example: *"38* (Figure 3f) *I counted up to 20. I counted as long as the time allowed"*. Or, *"20* (Figure 3a) *I counted 2, 4, 6 and arrived at 8, and I still had many dots left to count"*.

It seems that children using this strategy are using counting, which is a familiar and secure algorithm, as long as time allows. When they have to stop using it, they consider the *number (quantity) they already obtained* as a *frame of reference* to estimate the remaining quantity. This second step involves perceptual processes.

II. *Grouping strategy*. Children using this strategy, mentally divided the objects into groups, usually equal in number, which they then multiplied by the total number of groups. Counting or perceptual abilities were used to evaluate the small number of objects in each group. For example, *"24* (Figure 3b). *It had 6 on each side, and 6 times 4 is 24"*. Or, *"about 30* (Figure 3c) *The points are spread about 4 in each place. I circled groups of 4 in my head."* This is in agreement with the studies showing that subjects can perceptually evaluate the number of objects up to 8 (Folk et al., 1988). In this strategy the *size of the group* is used as an *external frame of reference.*

III. *Comparison strategy*. Children using this strategy compared the number of objects they observed to something they were familiar with.

They considered some familiar absolute information as an external frame of reference. For example, "50 (Figure 3c). It seems to me to have more than the two previous pictures". Or "20 (Figure 3b) I relied on the previous picture, and took off a few dots".

The comparison was more concrete with the necklace. "50. I have a similar necklace at home".

IV. *Global perception strategy*. It seems that children using this strategy could not explain how they arrived at their answers. They just glanced at the flash card and gave their estimate. For example, "50 (Figure 3f). I didn't count at all. It's according to the points. You can see it is 50".

Or the following, referring to Figure 3d, (C – child, I – interviewer). C: "25, I simply looked and guessed".

I: "You just guessed?"

C: "No, I guessed according to what I saw. I looked and guessed according to what I saw".

It seems that the strategy used by the children depends both on the individual differences between children and on the characteristics of the situation. Some geometrical arrangement of the dots led children to use the grouping strategy more than any other strategy; about 2/3 of the strategies used in Figures 3b and 3e, were grouping strategies. On the other hand, there were children who used the counting strategy for the necklace, even though it seemed to be very inefficient in that situation. Most of the children were not consistent in their responses and did not use the same strategy in all three pictures. However, three children used the counting strategy for all pictures in the first interview.

3.3. *Change in strategy*

Table I shows the changes in strategies used, as a result of going through the visual estimation unit. The percentages were calculated using the 36 explanations given for the three pictures by 12 children in each interview.

As can be seen, when comparing the two interviews, the most dramatic change in strategy was the decrease of the counting strategy (from 42% in the first interview, to 19% in the second) and the increase in the global perception strategy (from 16% in the first interview to 34% in the second). It appears that experience with the unit directed some of the children to abandon the counting strategy in favor of a more efficient strategy, espe-

TABLE I

Change in visual estimation strategies before and after exposure to the VEU. (Three pictures were shown to twelve children during each interview.)

Strategy	Interview prior to VEU	Interview following VEU
Counting	42%	19%
Grouping	31%	36%
Comparison	11%	11%
Global	16%	34%

cially in favor of the global perception strategy, which was used in 1/3 of the answers in the second interview. The following is an illustration for this change. One child explained in the first interview when presented with Figures 3 a and c, "I counted 2, 4, 6 up to 6". Or, "I started to count 2, 4, 6, and I reached 8". In the second interview the explanation of this child was quite different: "That's what I saw, I didn't start counting, I looked". Or, "There are many dots, it looks to me like there are 40".

Two children didn't change their counting strategies at all. They consistently used this strategy in the first interview, and continued to use it in the second as well.

The use of the two other strategies didn't change much from the first interview to the second. The grouping strategy increased from 31% to 36%, while the comparison strategy was 11% in both interviews.

We believe that there is a perceptual element underlying each strategy, where the difference is in the "size of the quantity" that was globally perceived. In the grouping strategy, for example, the children perceptually evaluated or estimated the group size and/or the number of groups. Frames of reference are used in counting, grouping and comparison strategies. In the first strategy the "counted quantity" was used as the frame of reference in estimating the remaining numbers. In the grouping strategy, the size of the group was used as the frame of reference, and in the comparison strategy, some external but familiar information was used as a frame of reference. In the grouping and comparison strategies, children chose to use frames of reference; however, in the counting strategy, the strategy itself, because of the shortage of time, forced them to rely on some frames of reference.

The above findings show that children are able to improve their ability to reach absolute visual information at a glance; the strategy they use becomes more efficient and is performed subconsciously. This finding is in

Absolute Error Situations

1) Noa and Gal were shown this dot picture for a short period of time and asked how many dots they saw. We know that there are 20 dots in the picture but the children, of course, didn't know it. Noa said that there are 24 dots, Gal said 26 dots. Did one of them give a better answer than the other or were both answers equally good?

2) A picture, with 30 dots, was shown to Noa and Gal. Gal said that there are 34 dots, Noa said 26. Same question as 1.

Relative Error Situations

3) Two dot pictures were shown to Noa; 20 in picture 1 (P1) and 50 in picture 2 (P2). Noa said 25 dots in P1 and 55 in P2. Was one of the answers better than the other, or were both answers equally good?

4) Two dot pictures were shown to Gal; 10 dots in P1 and 30 in P2. Gal said 15 in P1 and 40 in P2. Same question as in 3.

Visual Conflict Situations

5) Two dot pictures were shown to Noa; 10 in P1 and 100 in P2. Noa said that there are 11 in P1 and 102 in P2. Same question as in 3.

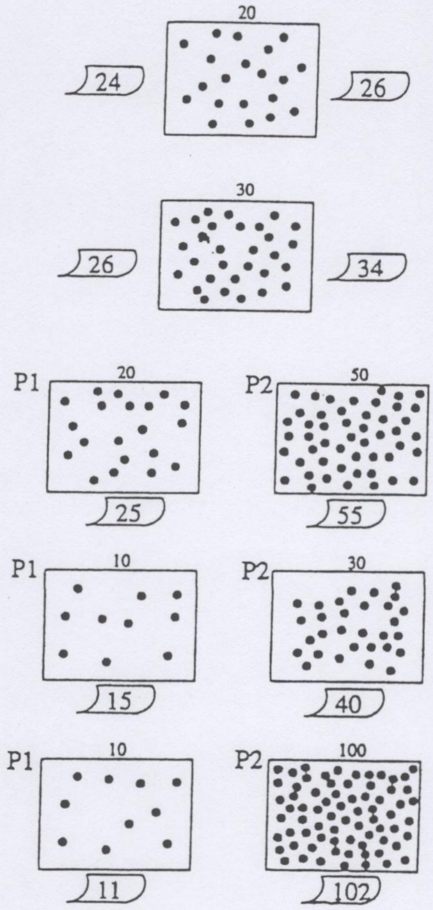


Figure 4. Situations presented at the first interview.

agreement with both Bryant's findings and theory, and the belief of Yaacov Agam as expressed in the goals of the visual estimation unit.

4. RESULTS – PART II: JUDGMENT PROCESSES IN VISUAL ESTIMATION (RESEARCH QUESTION C).

4.1. *The judgment situations*

To investigate students' awareness of the quality of the estimates, a set of judgment situations was developed. In both interviews, the situations depicted in Figures 4 and 5 were read to the child as the dot pictures were placed in front of him/her. Then the numbers given by Noa or Gal were put next to the picture as this information was read by the investigator. The

situations for the interviews, before and after the VEU, shown in Figures 4 and 5, were developed according to the following criteria:
i. Absolute error situations, (Judgment situations type 1; see section 1.2). Two different estimates of the *same* quantity were presented to the children. In these situations, judgment is based on the comparison of the absolute errors (1 and 2 , Figure 4).
ii. Relative error situations, (Judgment situations types 2 and 3; see section 1.2). The two estimates relate to *different* quantities. Thus, both the errors and the quantities to be estimated must be taken into account in order to arrive at the correct answer. Situation 3 (Figure 4) is the classic one where the absolute errors are the same for the two pictures; but because the quantities to be estimated are different, the relative errors are different. In situation 4 (Figure 4) both the quantities and errors differ in both pictures. This task is difficult, since the smaller absolute error turns to be relatively larger.
iii. Visual conflict situations. After interviewing four children, we noticed that children did not realize that since the quantities to be estimated were different, the errors had to be treated relative to the quantities. Therefore we added situation 5 (Figure 4) to the interview which we presented to the remaining eight children. In this situation we used two pictures where the difference between the quantities is clearly evident, both numerically and visually. We assumed that if the child could see the difference between the two pictures with his/her eyes in addition to seeing it with her/his mind's eyes, s/he would be pushed towards relative considerations.

In the second interview, which followed the VEU, we repeated situations 1–5 in Figure 4, with slight changes in the numbers depicted, and then the interview continued according to the following plan: If the child answered situation 5 correctly, the interview ended. If not, the child was presented with situations 6 & 7 in Figure 5, in which the intended conflict was even more aggravated, both in the numerical and visual sense. If the child did not succeed in either situations 6 or 7, then the interviewer intervened with situation 9, which we refer to as "the child from the other class situation", see Figure 5. Since we were not sure whether the aggravation of the visual conflict would work, we inserted this situation, in which considerations of easy/difficult in judgment, were first received legitimacy by another child, for the first time.

Situation 8 served as a check-up task, intended to be presented to children who succeeded in either or both situations 6 and 7 and to children who showed some kind of proportional reasoning in situation 9.

Aggravated Conflict Situations

6) Same situation as 5 except that Noa said 11 for P1 and 101 for P2

7) Two dot pictures were shown to Gal; 10 in P1 and 1000 in P2. Gal said 11 for P1 and 1001 for P2. Same question as in 3.

Check up Situation

8) Two dot pictures were shown to Noa; 5 in P1 and 100 in P2. Noa said 6 for P1 and 101 for P2. Same question as in 3.

The "Child from the other Class" Situation.

9) There was a child in another class who said, "It is true that in both pictures the error was 1, but in P1 there are 10 and in P2 1000 It is much more difficult to say 1001 when there are 1000, than to say 11 when there are 10, that's why Gal's answer for P2 is better." That's what the child said. What do you think?

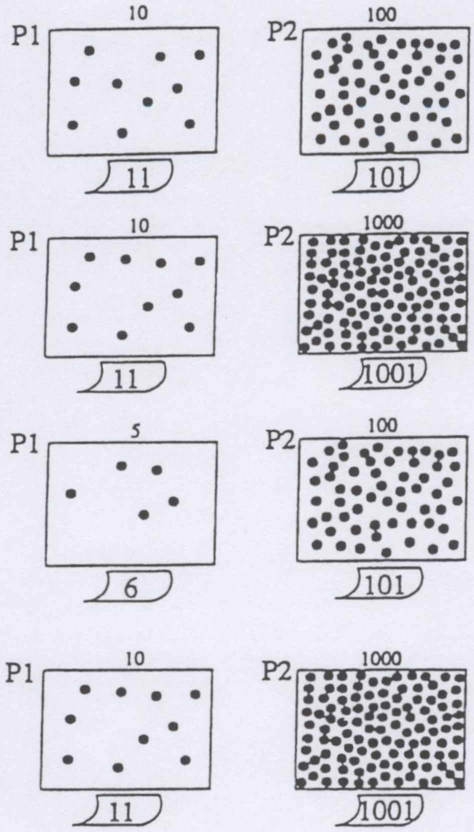


Figure 5. Situations presented at the second interview.

4.2. *Answers given by the children in the first interview (see Figure 4)*

Absolute error situations: Upon presentation of situations 1 and 2 in Figure 4, all the children showed some sort of additive behavior, which is appropriate in these situations. They either implicitly took into account the difference between the estimates and the quantities to be estimated, or explicitly considered the absolute errors. In situation 1, all said that Noa's answer was better because she was closer to the exact number of dots. One girl, Dana, claimed that Noa's and Gal's answers were the same, explaining that "Noa is more accurate, closer to 20, but Gal goofed only by 2 more, which is not so terrible in a situation like this." This unexpected answer, differed from those of the others and suggested that in estimation one does not have to be exact but has to be close enough.

In situation 2, ten children said that both answers were the same because "both were 4 dots away". One child said that Noa's answer was better, "because the number (meaning the estimate) is smaller, and the smaller the number, the better the answer". Another child said that Gal's answer was better because "it is in the same number as 30", which we interpreted as meaning that both Gal's answer and the exact number have the same digit in the ten's place.

Relative error situations: Upon presentation of situations 3 and 4, the children continued to think in terms of absolute errors, although the situation had been changed. They showed additive behavior in task 3 and said that both answers are the same. Calculations like "25 – 20=5, 55–50=5", were performed with the conclusion that "in both it is 5 more". In situation 4, they said that Gal's answer in P1 is better "because the error is only 5, and in P2 the error is 10".

Conflict situations: Only eight out of the twelve children were presented with situation 5 which had been added during the interview. Seven children said that: "P1 is better because it is closer to the exact number". Only Dana, the eighth child, showed a different behavior, saying that: "One of the answers is better. In P2, Noa is better, even though she missed by 2 and in P1, she missed only by 1. 102 is better, because in P1, it is easier". Dana combined the level of difficulty with the quality of the estimation process; she felt that more difficult to estimate, means a better estimate.

Following Dana's answer, we asked each of the other seven children in which situation, P1 or P2, is it more difficult to get closer to the exact number. A typical answer was: "When you have more dots". But the following was immediately added: "But it doesn't matter if you have more dots or less. What matters is the difference." Thus, it seems that these children did not have a conflict at all. They realized that although it was much more difficult to get a closer estimate to 100 dots, it had nothing to do with the question they were asked. They did not make the connection that Dana made between the difficulty to estimate and the quality of the estimate. One child seemed to change his answer at first, but then he resolved the conflict in the following way: "Maybe P2 is better because it is much more difficult. In principle, P1 is better and P2 is more difficult".

Since Dana was the only child who realized that the absolute error was not suitable to be used in this situation, we continued to question her and asked what her answer would be if Noa would say 11 dots for P1, but 105 dots for P2. Dana said that the estimate for P2 is still better. We asked what if the estimate for P1 was 11 and 110 for P2. Dana replied that she was not sure, because it is difficult since the numbers to be estimated are so different. "One number is around 10 and the other around 100. When Noa said 105, it was still all right, but suddenly when she had 10 more dots it looked different, as if it was raised by 20 dots. That's the way I decided". Dana wasn't explicitly using proportional reasoning, but that is what she was talking about implicitly. Moreover, she had a feeling for the numbers involved. She said that 102 and 105 are an acceptable estimate for 100, as compared to an estimate of 11 for 10 dots, but she was not sure that 110 was reasonable. Thus, Dana has "a sense of what 'looks right' or of what seems to be a distortion" (Hart, 1988, p. 217).

4.3. *Answers given by the children in the second interview, (Figure 5)*

The responses of most of the children to both the absolute error and relative error situations were very similar to their answers in the first interview. Children continued to use an additive approach and only looked at the difference between the estimate and the exact number of dots.

Only Dana responded differently using her "relative idea". For example, in situation 3, she said that: "55 is better, because it is more difficult when you have more dots".

In the *conflict situations,* all children (except Dana) did not have any conflict in situation 5 (Figure 4); thus, all of them were presented with tasks 6 and 7 (Figure 5) which emphasize the conflict situation. We hoped that when presented with two quantities that were very different (10 dots in P1 and 1000 dots in P2), while in both Gal missed only by 1, the children would move to proportional reasoning. However, only additive answers were given, "both estimates are the same, since he missed both by 1."

Dana again used her way of relative thinking and said that the estimate for P2 is better because "it is so much more difficult when you have more dots". We then presented her with tasks 6 and 7 (Figure 5) and in both she claimed that the estimate for P2 was better for the same reason she gave in situation 5 (Figure 4).

When we realized that the conflict situations did not push the children towards proportional reasoning, we presented them with situation 9 in Figure 5, in which we explicitly used Dana's idea of ease vs. difficulty in estimation. This approach did not have any impact upon five out of the eleven children. One even got angry and answered "It is the same, I already told you this a thousand times". The remaining six children, agreed that "the child from the other class" was right. They then were presented with the check up situation 8, in Figure 5. Two of the six adopted the following rule "whenever it is easier, the answer is better." This directly contradicts Dana's idea. The remaining four children, seemed to start thinking in proportional terms. One of them said: "The child in the other class is right. It is a disgrace to be wrong in P1. In P2, I could make the same mistake". Another did not let us finish the story with "the child from the other class", and immediately corrected his answer. All four, corrected their answers in situation 7, answered correctly 8, and corrected their answers for the previous situations as well.

As can be seen, children expressed an additive type of behavior in the relative error situations, in both interviews. They continued to relate only to the absolute errors in the conflict situations, even when the conflict was greatly aggravated. Dana was the only child who started to progress from this stage, as a consequence of the conflict situation. It seems that she had begun to develop a quantitative feeling about the ratios between the two variables involved in the situation presented. The children's additive approach seemed to start falling apart, for at least four of them, following the presentation of the situation of the "child from the other class". Some children moved towards the consideration of easy/difficult, which are quite natural when the quantities became quite big and on different scales.

5. Discussion

The goal of this study was to investigate *visual estimation* processes and processes of *judgment in visual estimation situations* of young children. These processes have strong visual characteristics which make them special and unique.

In the *visual estimation situations* (Part I), the strategies that children used are a combination of both; strategies of estimation processes and strategies of visual information processing. The main characteristics of this mixed process are the need that children had to provide absolute information about the estimated quantity on the one hand, and their limited ability on the other, because initially they could only gather information by relative codes. Our research findings show how the estimation processes progress between these two poles. Children develop strategies in which they can overcome the above limitation by the use of external frames of reference. The components involved in our research situations were of visual nature: the dots, their quantity, their arrangement, the frames of reference (which are based on other known quantities of dots) and the visual or imagery comparison of the quantity to be estimated with the frame of reference. Our data show that in the global perception strategy, this figural process is internalized so that the child could provide absolute visual estimates without being aware of how s/he made it. These data support those of Bryant (1974).

Most of the *judgment situations* (Part II), we used in this study, are *proportional reasoning* situations of comparison type. Most research on proportional reasoning has focused on adolescent subjects (Behr et al., 1992). In this study, we have investigated the reasoning of pre-adolescents around the ages of nine.

The situational elements of our proportional reasoning situations, have well-distinguished characteristics:

- Each ratio in the proportion expresses two different variables (the estimate or the error and the quantity to be estimated) which have the same measure (number of dots) and unit (dot);
- The absolute error calculation is an *additive* one. This may cause a delay in the development of the child's multiplicative reasoning;
- The size of the quantities (numbers) involved (A,B,C,D) are quite big, while in many classical situations the numbers are small;
- The two quantities to be estimated are of different order of magnitude (ranging from 10 to 1000 dots), while in the popular research situations, such as the example of Mr. Short and Mr. Tall (Karplus et al., 1974), the numerical values of the variable are of the same order;
- In addition, in our case the two ratios to be compared differ dramatically one from the other, in contrast to ratios involved in classical proportional tasks. This characteristic may emphasize the need to make the comparison between the ratios, rather than the comparison between one variable only, because the student may **see** and **feel** that 1 out of 10 is really different from 1 out of 1000.

It seems that the visual component which is integrated with all the above characteristics of the judgment situations, might stimulate the child, who is in an additive stage towards qualitative proportional reasoning, because he/she may insert considerations of easy/difficult.

In this study we investigated estimation and estimation judgment processes of discrete quantities. It will be interesting to investigate in parallel the same processes within the estimation of continuous quantities, like length or area. In addition, because the variety of responses, given by different children in our study, show evidence of different levels of judgment thinking which seems to lead towards proportional reasoning, it will be interesting to trace the development of proportional thinking processes in these kinds of situation over time.

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[1] *Center for Mathematics Education, Oranim School of Education Tivon, Israel*
[2] *Science Teaching Department, The Weizmann Institute of Science, Rehovot, Israel*