RESEARCH
INTO
PRACTICE

Conquer Mathematics Concepts by Developing Visual Thinking

It seems that visual thinking will be the primary way of thinking in the future. In modern life the presentation of phenomena has changed from tables and formulas heavy with numbers and symbols to a dynamic visual presentation using computers. To understand, analyze, and predict, we will have to engage in some form of visual thinking. To communicate visually and to develop advanced visual thinking, we need certain linguistic elements—a language.

The Agam program is an example of an effort to interweave the development of a visual language with a process of developing visual thinking. This project is the vision of the artist Yaacov Agam that has become an educational reality through the work of a team of researchers and educators of the Science Teaching Department in the Weizmann Institute. The program was developed, tested, and implemented with several groups of students beginning in nursery school with three-to-four-year-olds and continuing with the same groups to the third grade. The development and implementation was accompanied by research and evaluation that showed that the "Agam children" can apply visual abilities and visual thinking in learning tasks more successfully than children in control groups (Razel and Eylon 1990).

Some of the program's thirty-six curriculum units introduce students to such basic visual concepts as the main geometric figures, directions, colors, and size relationships. These units make up a "visual alphabet" that forms the basis for more advanced concepts, such as symmetry, ratio and proportion, numerical intuition, and other concepts that serve as building blocks in scientific and mathematical thinking.

The following example demonstrates how thinking is developed in the Agam program. The first unit is on the circle and the second on the square. The three-to-four-year-old student becomes acquainted with these concepts by methods relying almost exclusively on nonverbal experiences, for example, an activity in which each student takes one end of a rope (the ropes being first of equal lengths [see **fig. 1a**] and subsequently of different lengths [see **fig. 1b**]) whose other end is fixed to some point (the center) and by walking to create circles, experience visually and physically the properties of a circle.

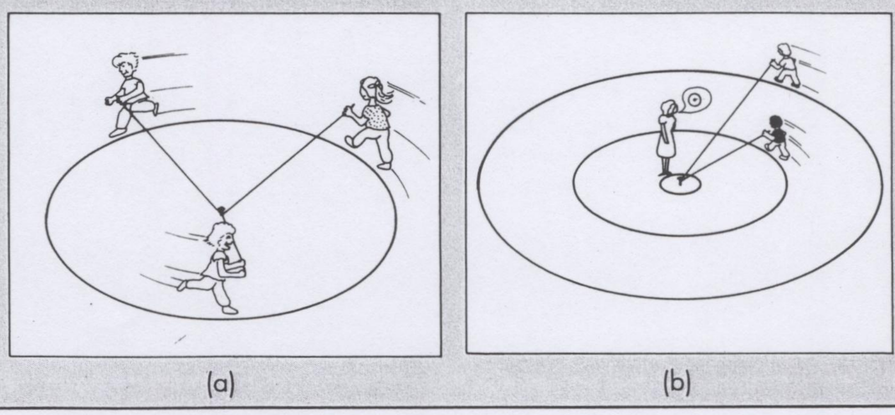
Visual thinking permeates modern life.

From circles and squares as "visual letters," patterns, which are "visual words" or "visual sentences," can be created (patterns are treated in the third unit in the program). Pattern, in the sense used here, is a visual periodic series whose elements at this stage in the curriculum are circles and squares of different sizes, colors, and position with changing intervals between them (see **fig. 2**). Periodic series are the bases of many mathematical and scientific concepts, such as certain functions, waves, and movements. This unit tackles the concept visually in a way that is meaningful to the young student.

When students identify patterns furnished by the teacher, books, or the classroom environment or when they memorize—store various patterns and re-

FIGURE 1

Students use a rope to make circles.



(a) (b)

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FIGURE 2

Patterns of squares and circles make words.

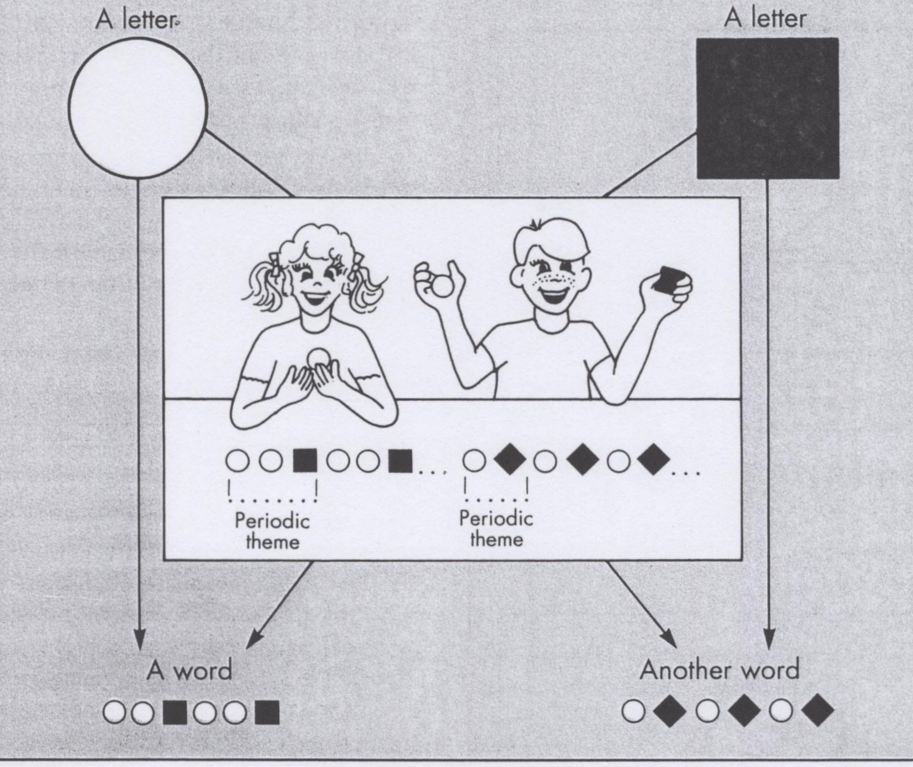
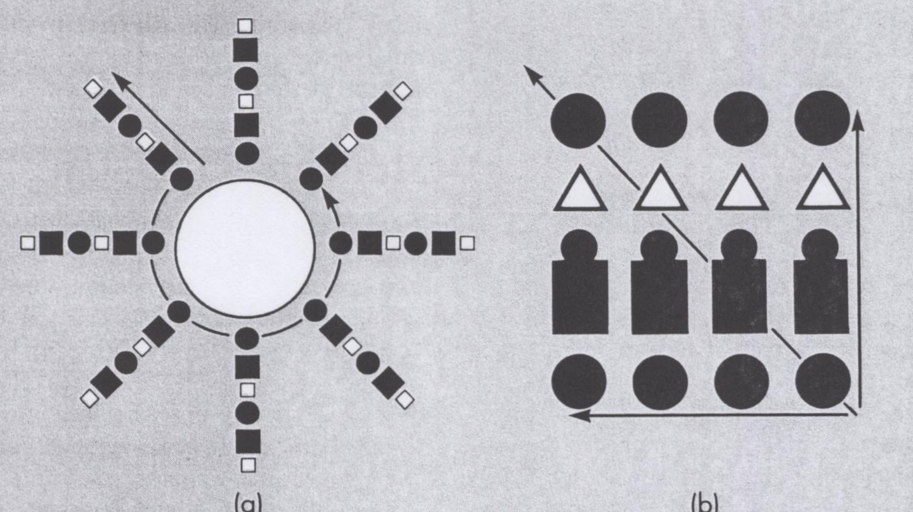


FIGURE 3

Multidimensional periodic patterns created by four-year-old students



call them—they internalize the concept of pattern and realize that it is the same irrespective of the changes in the periodic themes that create different patterns.

When students create patterns, they are problem solving with high-level thinking. They have to analyze the main characteristics of patterns—the building blocks that are to be used in their creation—and choose those that they would like to have in their own special pattern. Finally they have to synthesize all the aforementioned in the reproduction of their pattern. In the patterns unit all the activities deal with linear (one-dimensional) patterns, but children's creativity is unbounded. See, for instance, the "sun" in **figure 3a** in which the periodic series has a "ray" as the periodic theme and each "ray" itself is a linear pattern. The matrix (see **fig. 3b**) is also a multidimensional pattern: each row, each column, and the two main diagonals exhibit a linear pattern, and the whole matrix is a pattern with the column as the periodic theme.

In the following example we describe two units—"Ratio and Proportion" and "Numerical Intuition"—in which third graders, who have completed many units of the program, come closer to these higher-level mathematical concepts. These concepts, which are usually taught in higher grades, are the basis of many other concepts in mathematics, science, and other areas of life. Considerable evidence exists that these concepts are difficult for even adolescents to grasp (Tourniaire and Pulos 1985). Our experience indicates that children can overcome many difficulties if these concepts are experienced visually.

Ratio and Proportion

The following briefly describes the sequence of activities that are the basis of the visual presentation.

1. Sticks of various lengths are used, with a short one serving as a unit ("unit stick"). Students count how many times the unit can be placed beside two different sticks so as to equal their lengths and draw conclusions about the ratio between the lengths of these sticks.

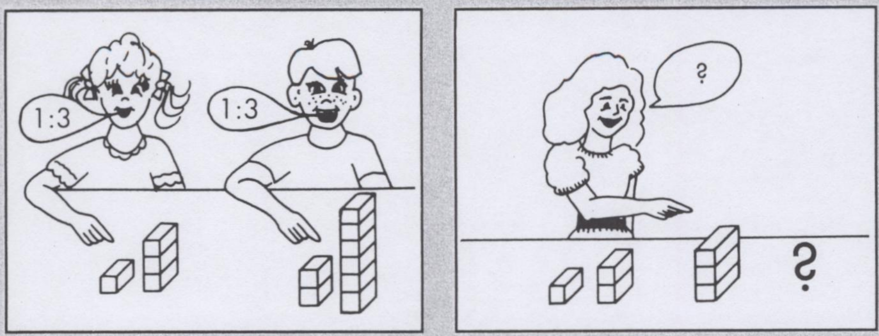
2. Repetition of the same activity with sticks of different lengths leads the students to discover a ratio between certain sticks regardless of their particular lengths.

3. Students are asked to find different pairs of sticks related in the same way. In this way they discover the concept of proportional relationship.

4. Students do similar activities with towers of blocks and colored liquid in glasses. Towers of various heights demonstrate that two ratios can be the same even if they are expressed by different numbers, for instance, 1/3 = 2/6 (see **fig. 4a**). This comparison establishes a visual basis for equality of fractions.

FIGURE 4

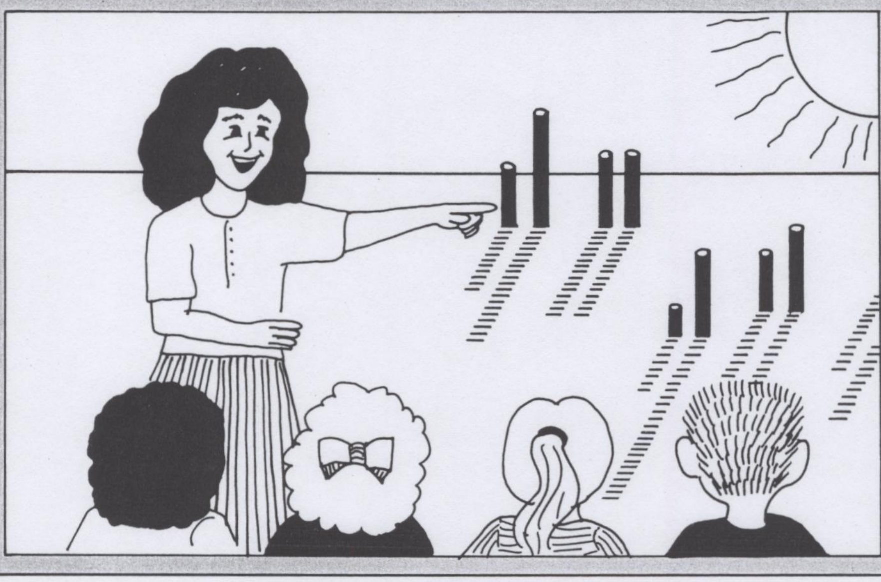
Students build pairs of block towers that are related in the same way.



(a) (b)

FIGURE 5

Students discover the proportion between the lengths of sticks and the lengths of their shadows.



5. Through similar activities with angles of different sizes (the unit on angle comes earlier in the program), students reinforce their intuition that the same ratios and proportions exist for different measures.

6. By measuring with sticks or paper strips, students find that the ratio of the lengths of the sides of similar shapes is constant.

7. Students discover that the ratio between the lengths of two vertical sticks is equal to the ratio between their shadows, thus extending their concept of proportion (see **fig. 5**).

8. In the more advanced activities three values in a proportion are given and the student is asked to find the fourth. For example, a pair of towers and a single tower are given; the student estimates the height of the fourth tower that completes the proportion and then checks the estimate by building the tower (see **fig. 4b**). In a similar activity the student estimates the height of a vertical stick given the height of a second stick and the length of the two shadows.

Classroom observations. As is to be expected, the level of learning varied among students and according to the type of task. The following are some examples of our observations of the students' learning.

• In activities 1, 2, and 3, the students leaned heavily on the unit sticks; for example, after they measured the lengths of two sticks to find the ratio, they would not remove the "unit sticks" even though they had already found the ratio. After time and with accumulated experiences most of them no longer needed the manipulative.

• In type-3 activities many students were satisfied with finding a single pair of sticks with a given ratio. With time, most of them succeeded in finding as many different pairs as could be obtained from the sticks at their disposal.

• Some students used trial-and-error strategies in finding a pair of sticks in a given ratio, whereas others planned their moves very carefully. For example, when finding a pair in the ratio 2 to 3, they first took five unit sticks, put two in a line and three in another line, and only then found suitable sticks.

• When presented with three sticks and asked to produce a drawing of sticks in the same ratios, some students explained their drawing qualitatively as big, medium, and small, whereas others succeeded in seeing and even expressing the ratios quantitatively.

• By the end of the unit, all the students were able to create different pairs of sticks, towers of blocks, and so on, for a given ratio (proportion), but few were able to deal successfully with activities in which three values in a proportion were given and the fourth one had to be found.

Numerical Intuition

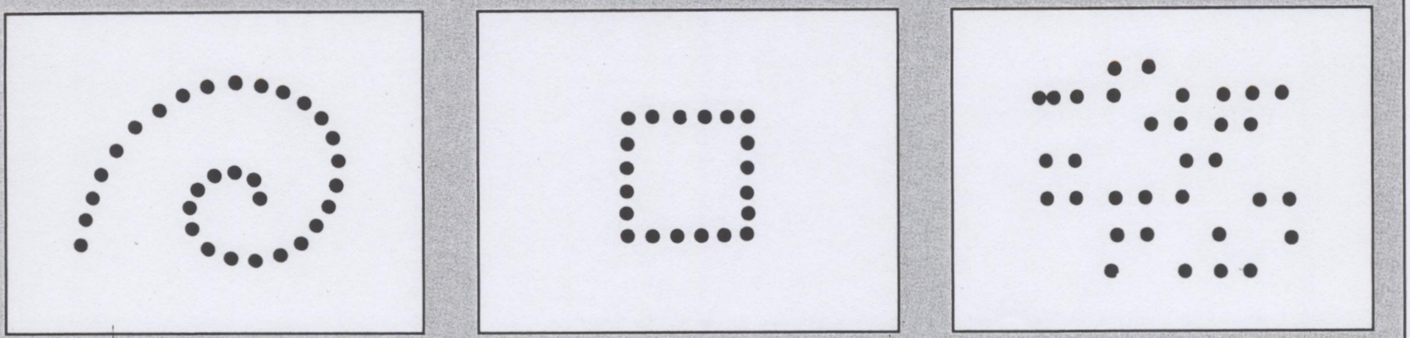
Numerical intuition is evidenced when a group of objects is shown briefly to students and they must determine the number of objects without counting. This activity occurs frequently in everyday life, but few of us bother to ask such questions as "How close was I to the exact number of objects?" "How did I get my number?" and "Do I use the same strategy each time?"

The following is a brief description of the activities in the numerical-intuition unit.

1. Students are presented with dot patterns (see **fig. 6**) for a very short time and then asked to write down how many dots they saw. As feedback the teacher allows them to count the exact number of dots and the students write the result next to their estimate.

FIGURE 6

Some patterns in the numerical-intuition unit



(a) (b) (c)

Students internalize the concept of pattern.

2. Students play a game that requires the winner not to reach an exact number but to come as close as possible to it. They experience being more or less close to the exact number.

3. Students perform number-estimation activities similar to the first one with such concrete objects as circles, squares, or pencils spread randomly on the desk; with pictures in children's books; and with objects from the classroom.

4. Students experience numerical-intuition activities outside the classroom.

Classroom observations. Students were interviewed before and after the unit. In the interview they were asked first to estimate the number of objects in a certain collection and then to explain how they obtained their estimate. The following strategies were identified:

1. Counting. In the short time allowed, some students counted as many objects as they could. This strategy is, of course, not very efficient.

2. Estimation. We found a few different estimation strategies:

- The student mentally divides the objects into groups with an equal number of objects in each. For example, in response to being shown **figure 6c,** Iair said, "Thirty—the points are spread, about four in each place. I circled groups of four in my head."
- Any other kind of estimation with a reasonable explanation, for example, Avner responded to **figure 6c,** "It's complicated, . . . twenty-five . . . , it's a bit more than the picture on the left.
- Some students used a kind of global estimation that does not use any algorithm or at least does not express the existence of such an algorithm, as demonstrated, for example, in responses like, "This is what it seems to me."

As a result of students' working through the unit, counting strategies dropped dramatically, whereas the use of all the aforementioned three estimation strategies increased, with the most dramatic increase in global estimation. We can conclude that the visual activities described developed the students' numerical intuition.

The activities in this unit lead directly to the question "How close am I to the exact number?" which is the basis of the concept of absolute and relative error. As in some ratio-and-proportion activities, students gave "additive" answers to the question, for example, that 11 dots is a better answer for a picture of 10 dots than 102 dots is for a picture of 100 dots. With a very short additional lesson, the students moved toward "relative" responses.

References

Razel, Micha, and Bat-sheva Eylon. "Development of Visual Cognition: Transfer Effects of the Agam Program." *Journal of Applied Developmental Psychology* 11 (1990):459–84.

Tourniaire, Francoise, and Steven Pulos. "Proportional Reasoning: A Review of the Literature." *Educational Studies in Mathematics* 16 (May 1985):181–204.

Photo Opportunity

The *Arithmetic Teacher* always has a need for good photographs for use on the cover. Photographs accepted for publication will give credit to the school and to the photographer. The following guidelines should be kept in mind when submitting sets of color slides for consideration: (*a*) The most effective pictures include one or two children; (*b*) backgrounds should not be too busy; (*c*) children should be actively engaged in a recognizable non-paper-and-pencil task; (*d*) the faces of the children should be clear; (*e*) children of different ethnic backgrounds should be included; and (*f*) racial, sexual, or ethnic stereotyping should be avoided. Release forms should be completed for each person pictured. Material should be sent to the Director of Publications, NCTM, 1906 Association Drive, Reston, VA 22091.